On Non-Archimedean frames

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Frames (locales, complete Heyting algebras) are complete lattices L such that the following distributivity holds:

$$a \wedge \left(\bigvee X\right) = \bigvee\{a \wedge x \mid x \in X\}$$

for each $a \in L$ and $X \subseteq L$.

Frames can be understood as an algebraic manifestation of a topological space. Indeed, for every topological space S, the lattice of open sets $\mathcal{O}S$ constitute a frame but not every frame is a topology. A decent analysis of the category of topological spaces can be done in the language of frames (see [PP12]). An example of the above statement is the core of this talk:

Definition. A *non-archimedean* topological space S is a Hausdorff space with a base $\mathcal{B}$ satisfying the *trichotomy* laws: If $B_1, B_2 \in \mathcal{B}$, we have that either $B_1 \cap B_2 = \varnothing$ or $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ holds (see, e.g., [Nyi75], [NR75], [Nyi99]).

Motivated by this, we introduce:

Definition. A frame is *non-archimedean* if it has a (non-archimedan) base $\mathcal{B}$ that satisfies these trichotomy laws: If $b_1, b_2 \in \mathcal{B}$, then either $b_1 \wedge b_2 = 0$ or $b_1 \leq b_2$ or $b_2 \leq b_1$ holds.

One of the main examples of non-archimedian frames comes from non-archimedian fields, by example consider the frame of p-adic numbers $\mathcal{L}(\mathbb{Q}_p)$, defined by [Ávi20], where $\mathbb{Q}_p$ is the field of p-adic numbers. This field was determined by Kurt Hensel in 1904 in analogy with the Laurent series $\mathbb{C}((t))$. Furthermore, this field is a non-archimedean field with the p-adic norm $|\cdot|_p$ defined over it. Since the set of open balls centered at rationals generates the open subsets of $\mathbb{Q}_p$, we consider these balls' (lattice) properties and think of them as generators. Thus, we can define the $\mathbb{Q}_p$ frame as follows.

Let $\mathcal{L}(\mathbb{Q}_p)$ be the frame generated by the elements $B_r(a)$, where $a \in \mathbb{Q}$ and $r \in |\mathbb{Q}| := \{p^{-n}, n \in \mathbb{Z}\}$, subject to the following relations:

(1) $B_r(a) \wedge B_s(b) = 0$ whenever $|a-b|_p \geq r \vee s$.

(2) $1 = \bigvee\{B_r(a) : a \in \mathbb{Q}, r \in |\mathbb{Q}|\}$.

(3) $B_r(a) = \bigvee\{B_s(b) \colon |a-b|_p < r, s < r, r \in |\mathbb{Q}|\}$.

Note that relation (3) implies that the set

$$\mathcal{B} := \{B_r(a) : r \in \mid \mathbb{Q}|, a \in \mathbb{Q}\}$$

is a base for $\mathcal{L}(\mathbb{Q}_p)$. Moreover, let $B_r(a)$, $B_s(b)$ be any two elements in $\mathcal{L}(\mathbb{Q}_p)$ and, without loss of generality, assume that $s \leq r$. Then, If $|a-b|_p \geq r$, we have $B_r(a) \wedge B_s(b) = 0$ by relation (1), and if $|a-b|_p < r$, we have $B_s(b) \leq B_r(a)$ by relation (3). Thus, for any $B_r(a), B_s(b) \in \mathcal{B}$, either

$$B_r(a) \wedge B_s(b) = 0 \text{ or } B_s(b) \leq B_r(a) \text{ or } B_s(b) \geq B_r(a).$$

It follows that $\mathcal{B} := \{B_r(a) : r \in |\mathbb{Q}|, a \in \mathbb{Q}\}$ is a non-archimedean base for the frame $\mathcal{L}(\mathbb{Q}_p)$, and that $\mathcal{L}(\mathbb{Q}_p)$ is a non-archimedean frame.

Another example of a similar nature of a non-archimedean frame is the frame of the Cantor set, denoted by $\mathcal{L}(\mathbb{Z}_p)$ [ÁUZ22].

As the examples show, the bases of these non-archimedean frames constitute a tree. This phenomenon is not a coincidence; in [Nyi99, Theorem 2.10] the author shows that every non-archimedean space is a subspace of a branch space of a tree. In the point-free context we have:

Theorem. Let A be a non-archimedean frame with base $\mathcal{B}$. Then A has a tree-base.

Since for every tree we have its branch space we have furthermore:

Theorem. A frame A is non-archimedean if and only if A is a quotient of a topology of a branch space of a tree.

In this talk we will give some details on these theorems and their connection with the spatiality of certain quotients of the Alexandroff topology given by the tree-base.

References

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