Interpolation in Some Modal Substructural Logics

Wesley Fussner[1] and Simon Santschi[2]

[1] Institute of Computer Science, Czech Academy of Sciences, Czechia
fussner@cs.cas.cz
[2] Mathematical Institute, University of Bern, Switzerland
simon.santschi@unibe.ch

The *deductive interpolation property* stipulates that, for the consequence relation $\vdash$ of some propositional logic and any set of formulas $\Gamma \cup \{\phi\}$,

> if $\Gamma \vdash \phi$, then there exists a set of formulas Γ' whose variables are among those contained in both Γ and ϕ such that $\Gamma \vdash \Gamma'$ and $\Gamma' \vdash \phi$.

Deductive interpolation has been studied in a range of different contexts. Famously, Maksimova showed in [7] that, of the continuum-many consistent superintuitionistic logics, only 7 have the deductive interpolation property (equivalent in that context to the better known Craig interpolation property). Later, in [8], she showed that there are at most 49 consistent normal extensions of the modal logic **S4** with the deductive interpolation property.

In this work, we study deductive interpolation in a substructural environment that combines these. In particular, we examine expansions of substructural logics with the exchange rule—but possibly lacking the contraction or weakening rules—by **S4**-like modalities. Our work proceeds by first exhibiting continuum-many axiomatic extensions of the full Lambek calculus with exchange $\mathbf{FL_e}$ that have the deductive interpolation property, and then showing that—because of the special form of the extensions constructed—each of these may be expanded by an **S4**-like modality. In this fashion, we also obtain continuum-many **S4**-like modal expansions of $\mathbf{FL_e}$ that have deductive interpolation. Previously, only countably many substructural logics were known to have the deductive interpolation property, so our work contributes to the general theory of interpolation in substructural logics as well as stands in contrast to Maksimova's results.

For suitable algebraizable logics, there is a well-known connection between the deductive interpolation property for a logic and the amalgamation property for its associated class of algebraic models (see, e.g., [2]). Consequently, much like Maksimova, our work focuses on the study of amalgamation in appropriately chosen algebraic models. The fundamental algebraic structures we consider are FL_e*-algebras*, i.e., algebras of the form $\langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a commutative monoid, and $x \cdot y \leq z$ if and only if $x \leq y \to z$. To treat modalities, we define an $S4FL_e$*-algebra* to be an expansion of an FL_e-algebra by an additional unary operation $\Box$ that satisfies the identities

1. $\Box(x \wedge y) = \Box x \cdot \Box y$.
2. $\Box\Box x = \Box x \leq x$.
3. $\Box 1 = 1$.

To find continuum-many logics with deductive interpolation, we construct continuum-many varieties of FL_e-algebras with the amalgamation property (see [3] for relevant definitions). These varieties are constructed by first considering suitably chosen quasivarieties of abelian groups, each with the amalgamation property. The abelian groups contained in these quasivarieties are then transformed into FL_e-algebras using a construction that preserves the amalgamation property, and the latter are used as generating algebras for the varieties we are interested in. The examples we construct are sufficiently transparent to lift the amalgamation property from the generating algebras using existing tools (see, e.g., [3, 9]), but also sufficiently flexible that they may be expanded into $S4FL_e$-algebras so as to keep the amalgamation property. Thus:

Theorem 1.

1. *There are continuum-many varieties of FL_e-algebras that have the amalgamation property*
2. *There are continuum-many varieties of $S4FL_e$-algebras that have the amalgamation property.*

FL_e-algebras are well-known to algebraize the consequence relation of the full Lambek equipped with the exchange rule [5]. Likewise, $S4FL_e$-algebras algebraize an **S4**-like modal variant of the full Lambek calculus with exchange, which we call $\mathbf{S4FL_e}$. The logics algebraized by the varieties considered in Theorem 1 all enjoy local deduction theorems, so by well-known bridge theorems linking amalgamation and deductive interpolation we obtain the following:

Theorem 2.

1. *There are continuum-many axiomatic extensions of* $\mathbf{FL_e}$ *that have the deductive interpolation property.*
2. *There are continuum-many axiomatic extensions of* $\mathbf{S4FL_e}$ *that have the deductive interpolation property.*

The techniques we use to obtain Theorems 1 and 2 are extremely flexible, and also allow us to obtain a host of similar results for related logics. Of these, we mention only Girard's celebrated linear logic [6], which is algebraized by certain expansions of $S4FL_e$-algebras (see [1]):

Theorem 3. *Classical linear logic has continuum-many axiomatic extensions with the deductive interpolation property.*

Further information on this work may be found in our preprint [4].

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