Subdirectly irreducible and generic equational states

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An Abelian lattice-ordered group (ℓ-group, for short) is an Abelian group G endowed with a lattice order that is translation invariant. An ℓ-group is called unital if it contains an element u, such that for any positive $g \in G$ there exists a natural number n for which the n-fold sum of u exceeds g. A *state* of a unital ℓ-group is a normalized and positive group homomorphism in $\mathbb{R}$. It is well known that states correspond to expected-value operators on bounded real random variables. Unital ℓ-groups are not first-order definable, yet they are categorically equivalent to the equational variety of MV-algebras [1]. Thus, states can be studied in an equational setting by looking at their counterpart in MV-algebras, as first proposed in [9]. However, since states on MV-algebras are defined as particular maps into the real unit interval $[0, 1]$, a completely algebraic characterization was still missing.

Efforts to find an algebraic theory of states continued in [5] (see also [2]). There the authors introduced the notion of *internal state* as an additional unary operation with specific axioms relating it to the other MV-operations. This framework was used to provide an algebraic treatment of the Lebesgue integral. A drawback of this approach is that an internal state can be applied to itself. More recently, a different approach has been proposed. In [6] the authors first extend Mundici's equivalence between unital ℓ-groups and MV-algebras to an equivalence between states between ℓ-groups and states between MV-algebras. Secondly, they introduce the class of *equational states* as a two-sorted variety of algebras. An equational state $(\mathbf{A}_1, \mathbf{A}_2, s)$ is a two-sorted algebra in which each sort $\mathbf{A}_1$ and $\mathbf{A}_2$ is an MV-algebra with customary operations and the *state-operation* s has $\mathbf{A}_1$ as domain and $\mathbf{A}_2$ as codomain. This approach opens the way to studying probabilistic notions with algebraic tools; for instance, [6, Theorem 4.1] gives a characterization of free equational states.

Another reason for considering the class of equational states is that they provide an algebraic semantics to the probabilistic logic FP(Ł, Ł). The system FP(Ł, Ł) is a two-layer logic introduced in [4] to provide a formal framework to deal with the probability of vague events. If a vague event is codified by a formula φ in Łukasiewicz logic, its probability is given by the formula $\square(\varphi)$, which is a Łukasiewicz atomic formula interpreted as "φ is probable".

An adaptation of the classical Lindenbaum-Tarski construction produces an equational state $\mathcal{ES}_{\mathsf{Var}}$ with the following properties.

Theorem 1 ([7, Theorem 8]). *Let Var be a (one-sorted) set of propositional variables. For any FP(L, L) formula Φ, the following are equivalent.*

1. *Φ is a theorem.*
2. *Φ is valid in the equational state $\mathcal{ES}_{\mathsf{Var}}$.*
3. *Φ is valid in all equational states.*

Corollary 1 ([7, Theorem 15]). *The equational state $\mathcal{ES}_{\mathsf{Var}}$ is the free equational state generated by $(\mathsf{Var}, \emptyset)$.*

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We present here a continuation of the algebraic study of equational states started in [8], where it is proven that the lattice of ideals and the lattice of congruences of any equational state are isomorphic (see [8, Corollary 2]). This isomorphism enables us to characterize the subdirectly irreducible equational states as follows.

Theorem 2. *An equational state* $(\mathbf{A}_1, \mathbf{A}_2, s)$ *is subdirectly irreducible if and only if one of the following is true:*

1. $\mathbf{A}_2 = \emptyset$ *and* $\mathbf{A}_1$ *is a subdirectly irreducible MV-algebra.*
2. $\mathbf{A}_2$ *is a subdirectly irreducible MV-algebra, and the state-operation is faithful, i.e.* $s(x) = 0$ *implies* $x = 0$.

Combining the characterization of subdirectly irreducible equational states with some ideas of [3] we prove that two notable classes generate the variety of equational classes.

Theorem 3. *The following classes generate the variety of equational states:*

1. *The class of all equational states of the type* $([0,1]^W, [0,1])$*, with* W *an arbitrary set.*
2. *The class of finite equational states, i.e. equational states whose universe is finite in each sort.*

References

[1] Roberto L. O. Cignoli and Itala M. L. D'Ottaviano and Daniele Mundici, *Algebraic Foundations of Many-Valued Reasoning*, Kluwer Academic Publishers, 2000.

[2] Antonio Di Nola and Anatolij Dvurečenskij, *State-morphism MV-algebras*, Annals of Pure and Applied Logic, 161(2), 2009, 161-173.

[3] Tommaso Flaminio, *On stardard completeness and finite model property for a probabilistic logic on Łukasiewicz events*, International Journal of Approximate Reasoning, 131, 2021, 136–150.

[4] Flaminio, Tommaso and Godo, Lluís, *A logic for reasoning about the probability of fuzzy events*, Fuzzy Sets and Systems, 158 (6), 2007, 625–638.

[5] Tommaso Flaminio and Franco Montagna, *MV-algebras with internal states and probabilistic fuzzy logics*, International Journal of Approximate Reasoning, 50(1), 2009, 138–152.

[6] Tomas Kroupa and Vincenzo Marra, *The two-sorted algebraic theory of states, and the universal states of MV-algebras*, Journal of Pure and Applied Algebra, 225, 2021.

[7] Serafina Lapenta, Sebastiano Napolitano and Luca Spada, *The logic FP(Ł,Ł) and two-sorted equational states*, International Symposium on Imprecise Probability: Theories and Applications, in PMLR 290, 2023, 280–287.

[8] Serafina Lapenta, Sebastiano Napolitano and Luca Spada, *Ideals in the Two-Sorted Variety of Equational States*,Conference of the European Society for Fuzzy Logic and Technology, Springer, 2023, 495–504.

[9] Daniele Mundici, *Averaging the truth-value in Łukasiewicz logic*, Studia Logica, 1, 1995, 113–127.