Notes on ω-well-filtered spaces

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Abstract

We prove that a T_0 topological space is ω-well-filtered if and only if it does not admit either the natural numbers with the cofinite topology or with the Scott topology as its Skula closed subsets. Based on this, we offer a refined topological characterization for the ω-well-filterification of T_0-spaces and solve a problem of Xu.

In Mathematics, various types of "completions" of different structures have been drawing extensive attention and acting as important roles both in theory and practice. Examples include the Dedekind-MacNeille completion of ordered sets, completion of metrics and compactifications of topological spaces, et cetera. In domain theory and non-Hausdorff topology, completions such as the D-completion [6], well-filterification [9] and sobrification [3] of T_0 spaces are particularly well-studied in the form of reflectivity of the corresponding categories, to name a few.

The notion of ω-well-filtered spaces, which is strictly weaker than that of well-filtered spaces (hence that of sober spaces) introduced by Heckmann [5], is initially put forward by Xu et al. [10].

Definition 0.1. A T_0 space X is called *ω-well-filtered* if for every reversely ordered countably family $\{K_i\}_{i\in\mathbb{N}}$ of compact saturated subsets ($K_i \subseteq K_j$ when $i \geq j$), that the intersection $\bigcap_{i\in\mathbb{N}} K_i$ is in some open subset U implies that $K_i \subseteq U$ for some $i \in \mathbb{N}$.

Examples of ω-well-filtered spaces include all well-filtered spaces hence all sober spaces. Like well-filtered spaces and sober spaces, ω-well-filtered spaces have many nice properties. For example, the classical result that a sober space is locally compact if and only if it is core-compact can be extended to ω-well-filtered spaces. Xu et al. [10] showed that the category of all ω-well-filtered spaces is a reflective full subcategory of the category of T_0 spaces with continuous maps. This reveals the existence of "ω-well-filterification" for T_0 spaces. In the same paper, they gave a direct characterization of this completion by identifying the corresponding completion space as the family of all WD_ω-subsets endowed with the lower Vietoris topology.

In this paper, we look more closely at ω-well-filtered spaces through the lens of descriptive set theory. Recently, de Brecht obtained that a countably based T_0 space is sober if and only if it does not contain a Π^0_2-subspace homeomorphic to one of two specific topological spaces S_1 or S_D [2], where S_1 and S_D are the natural numbers with the co-finite topology and the Scott topology (in the usual order), respectively. This result was generalized to first-countable T_0 spaces in [7], and the authors showed that a first-countable T_0 space is sober if and only if it does not contain a Π^0_2-subspace homeomorphic to S_1, S_D or a directed subset without a maximum element. In a similar but different vein, and as a central result of this paper we prove:

Theorem 0.2. *A T_0 space is ω-well-filtered if and only if it does not contain S_1 or S_D as its Skula closed subsets.*

This provides characterizations for ω-well-filtered spaces by forbidden subspaces. In [10], Xu et al. also proved that on each first-countable T_0 space, well-filteredness and sobriety coincide. A natural question related to this matter is whether the well-filterification and sobrification constructions also coincide on first-countable T_0 spaces, which boils down to proving whether every first-countable T_0 space is a well-filtered determined space in the sense of Xu et al. [11]. The authors of [11] further posed the following problem:

Problem 0.3. Is every first-countable T_0 space a Rudin space?

Based on our aforementioned characterization for ω-well-filtered spaces, we solve Problem 0.3 in the negative by displaying a counterexample.

Moreover, our characterization for ω-well-filtered spaces via forbidden subspaces enables us to give more refined characterizations for the D-completion of Keimel and Lawson and also for the sobrification, when the underlying space is second-countable. This is achieved via the aid of a weaker version of the strong/Skula topology, which we introduce in this paper.

Definition 0.4 (Strong$_*$ topology)**.** Let X be a T_0 space. A nonempty subset A of X is said to have the *KF_ω property*, if there exists a countable filtered family $\mathcal{K}$ of compact saturated sets of X such that $\mathrm{cl}(A)$ is a minimal closed set that intersects all members of $\mathcal{K}$.
Let $\mathcal{B} = \{A \subseteq X \mid \sup B \in A \text{ for all } B \in \mathrm{KF}_\omega(A) \text{ with } \sup B \text{ existing}\}$. Then the family $\mathcal{B}$, as closed sets, forms the *strong$_*$ topology* of X.

Theorem 0.5.
1. *In each ω-well-filtered space, all of its ω-well-filtered subspaces are precisely its closed subsets in the Strong$_*$ topology.*
2. *The ω-well-filterification of a T_0-space X is homeomorphic to the Strong$_*$ closure of the embedding copy of X in the sobrification of X.*

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