# Non-distributive description logics *

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*Basic non-distributive modal logic* (a.k.a. *LE-logic*) is the non-distributive counterpart of positive modal logic without the distributivity axiom. Algebraically, it can be viewed as the logic of arbitrary lattices expanded with normal modal operators. Polarity-based semantics for LE-logic is given by a tuple $M = (\mathbb{F}, V)$, where $\mathbb{F} = (A, X, I, R_\Box, R_\Diamond)$ is an *enriched formal context* [3], i.e. , a *formal context* $\mathbb{P} = (A, X, I)$ enriched with *I-compatible* [3] relations $R_\Box \subseteq A \times X$ and $R_\Diamond \subseteq X \times A$, and $V$ is a valuation map which maps LE-formulas to *formal concepts* defined by $\mathbb{P}$. Due to its natural connection with Formal Concept Analysis [4], LE-logic with its polarity-based semantics has been studied as the "logic of categorization" expanded with modal operators [3]. Motivated by this insight, in [5] we defined a two-sorted non-distributive *description logic* counterpart of LE-logic called LE-$\mathcal{ALC}$.

LE-$\mathcal{ALC}$ provides a natural description logic [1] to represent and reason about (partial) knowledge about formal contexts and concepts defined by them. LE-$\mathcal{ALC}$ has same concept names as LE-logic formulas, and has an analogous intended interpretation on the complex algebras of enriched formal contexts. This is similar to the classical case, where concept names of description logic are same as $\mathcal{ALC}$ and are interpreted over Kripke semantics in a similar manner.

Concept names in LE-$\mathcal{ALC}$ over a set of atomic concepts $\mathcal{D}$ are defined as follows:

$$C := D \in \mathcal{D} \mid C \wedge C \mid C \vee C \mid \top \mid \bot \mid \langle R_\Diamond \rangle C \mid [R_\Box]C,$$

As usual, $\vee$ and $\wedge$ are to be interpreted as the smallest common superconcept and the greatest common subconcept. The constants $\top$ and $\bot$ are to be interpreted as the largest and the smallest concept, respectively. Like in the classical case, modal operators can be assigned various interpretations such as knowledge or approximation [3, 2]. LE-$\mathcal{ALC}$ has individual names of two types $\mathsf{OBJ}$ and $\mathsf{FEAT}$ intended to be interpreted as object and features names, respectively. LE-$\mathcal{ALC}$ ABox assertions are of the form:

$$aR_\Box x, \quad xR_\Diamond a, \quad aIx, \quad a : C, \quad x :: C, \quad \neg\alpha,$$

where $\alpha$ is any of the first five ABox terms, and TBox assertions are of the form $C_1 \equiv C_2$ for two concept names $C_1$ and $C_2$. The intended interpretation of term $a : C$ (resp. $x :: C$) is object (resp. feature) $a$ (resp. $x$) is an element of (resp. feature describing) $C$. Relational terms are interpreted in natural manner, and term $\neg\alpha$ is as negation of term $\alpha$. Term $C_1 \equiv C_2$ is interpreted as concepts $C_1$ and $C_2$ are equivalent.

An *interpretation* for LE-$\mathcal{ALC}$ is a tuple $\mathrm{I} = (\mathbb{F}, \cdot^{\mathrm{I}})$, where $\mathbb{F} = (\mathbb{P}, \mathcal{R}_\Box, \mathcal{R}_\Diamond)$ is an enriched formal context, and $\cdot^{\mathrm{I}}$ maps:

1. individual names $a \in \mathsf{OBJ}$ (resp. $x \in \mathsf{FEAT}$), to some $a^{\mathrm{I}} \in A$ (resp. $x^{\mathrm{I}} \in X$);
2. relation names $I$, $R_\Box$ and $R_\Diamond$ to relations $I^{\mathrm{I}}$, $R_\Box^{\mathrm{I}}$ and $R_\Diamond^{\mathrm{I}}$ in $\mathbb{F}$;
3. any primitive concept $D$ to $D^{\mathrm{I}} \in \mathbb{F}^+$, and other concepts as follows:

$$\bot^{\mathrm{I}} = (X^{\downarrow}, X) \qquad \top^{\mathrm{I}} = (A, A^{\uparrow}) \qquad (C_1 \wedge C_2)^{\mathrm{I}} = C_1^{\mathrm{I}} \wedge C_2^{\mathrm{I}}$$
$$(C_1 \vee C_2)^{\mathrm{I}} = C_1^{\mathrm{I}} \vee C_2^{\mathrm{I}} \qquad ([R_\Box]C)^{\mathrm{I}} = [R_\Box^{\mathrm{I}}]C^{\mathrm{I}} \qquad (\langle R_\Diamond \rangle C)^{\mathrm{I}} = \langle R_\Diamond^{\mathrm{I}} \rangle C^{\mathrm{I}}$$

*Krishna Manoorkar is supported by the NWO grant KIVI.2019.001.

An interpretation I is a *model* for an LE-$\mathcal{ALC}$ knowledge base $(\mathcal{A}, \mathcal{T})$ if $\mathrm{I} \models \mathcal{A}$ and $\mathrm{I} \models \mathcal{T}$. In [5], we proved the following theorem regarding the complexity of checking consistency of LE-$\mathcal{ALC}$ knowledge bases.

**Theorem 1.** *A tableaux algorithm exists for LE-$\mathcal{ALC}$, offering a sound and complete polynomial time decision procedure for verifying the consistency of LE-$\mathcal{ALC}$ knowledge bases by constructing a polynomial size model $Tab(\mathcal{K})$ for any consistent knowledge base.*

Several extensions of $\mathcal{ALC}$ with different concept constructors and axioms have been extensively researched. On our ongoing work, we generalized these results to extension of LE-$\mathcal{ALC}$ with axioms reflexivity, symmetery, and transitivity called LE-$\mathcal{ALCR}$ which can be seen as description logic for *rough concepts* [2]. We also proved similar results for extension of LE-$\mathcal{ALCR}$ with two new constructors: feature inconsistency pairs (i.e., pairs of features that no object can share) and concepts generated by sets of features.

Description logic ontologies play a crucial role in providing answers to queries based on incomplete databases. The following property of the model constructed by the Tableaux algorithm for LE-$\mathcal{ALC}$ is crucial with regards to querry answering over LE-$\mathcal{ALC}$.

**Lemma 1.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ be a consistent LE-$\mathcal{ALC}$ knowledge base with acyclic TBox. Let $b$, $y$, $C$, and $C'$ be any concept names appearing in $\mathcal{T}$. Then for any term $t$ consisting of individual, role, and concept names appearing in $\mathcal{K}$,*
*$Tab(\mathcal{T}) \models t$ iff for every model I of $\mathcal{T}$, $\mathrm{I} \models t$ .*

Lemma 1 implies that many querries over LE-$\mathcal{ALC}$ knowledge bases like **ascription querries** ('does object $b$ has feature $y$', 'name all the objects having feature $y$', etc. ), **membership querries** ('does object $b$ belong to concept $C$', 'name all the features defining concept $C$', etc. ), **subsumption querries** ('Is concept $C_1$ included in $C_2$'?, 'Name all the concepts included in $C_1$') can be answered by only looking at the model $Tab(\mathcal{K})$. As $Tab(\mathcal{K})$ can be constructed in polynomial time and is of polynomial size (in size of $|\mathcal{K}|$), we can answer querries over LE-$\mathcal{ALC}$ knowledge bases with acyclic TBoxes in polynomial time.

We believe that similar approach can be used to answer more complex querries like *ontology equivalence querries* ('Are two given ontologies equivalent?') and to perform tasks like *querry-based ontology learning* in polynomial-time. We believe these results show that LE-$\mathcal{ALC}$ and its extensions allows us to solve many important reasoning tasks relating to knowledge representation and reasoning in relation to formal contexts and concepts efficiently.

## References

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