On Free Generalized 3-valued Post algebras

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Abstract

We develop the theory of generalized 3-valued Post algebras (P_3^ω-algebras), which are obtained from Komori type S_2^ω-algebras by enriching its signature with the constant $\frac{1}{2}$. Finitely generated free and projective algebras are described in the variety $\mathbf{P}_3^\omega$ generated by P_3^ω-algebras. The variety $\mathbf{P}_3^\omega$ contains only one proper subvariety – the subvariety of 3-valued Post algebras [5].

1 P_3^ω-algebras

We introduce new class $\mathbf{P}_3^\omega$ of generalized 3-valued Post algebras that form a variety. P_3^ω-algebra is a system $(A, \vee, \wedge, \oplus, \otimes, \neg, 0, \frac{1}{2}, 1)$, where A is a nonempty set of elements, 0, $\frac{1}{2}$, and 1 are distinct constant elements of A, and $\vee, \wedge, \oplus, \otimes$ are binary operations on elements of A, and $\neg$ is a unary operation on elements of A, obeying a finite set of axioms (identities).

The algebra $(A, \vee, \wedge, \oplus, \otimes, \neg, 0, \frac{1}{2}, 1)$ is P_3^ω-algebra if $(A, \oplus, \otimes, \neg, 0, 1)$ is an S_2^ω-algebra (defined by Komori in [3]), i. e. MV-algebra satisfying the identity $(3(x^2))^2 = 2(x^3)$, and $(A, \vee, \wedge, 0, 1)$ is a distributive bounded lattice satisfying the following identities: $\frac{1}{2} \oplus \frac{1}{2} = 1$, $\frac{1}{2} \otimes \frac{1}{2} = 0$, $\frac{1}{2} \otimes (x \wedge \neg x) = 0$, $\frac{1}{2} \oplus (x \vee \neg x) = 1$, $\neg \frac{1}{2} = \frac{1}{2}$. The algebra $(\{0, \frac{1}{2}, 1\}, \vee, \wedge, \oplus, \otimes, \neg, 0, \frac{1}{2}, 1)$ with the following operations: $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$, $x \oplus y = \min(1, x + y)$, $x \otimes y = \max(0, x + y - 1)$, $\neg x = 1 - x$, is an example of 3-valued Post algebra. Notice, that this algebra is obtained by enriching the signature of an MV_3-algebra S_2 [2] with the constant $\frac{1}{2}$. Moreover, the algebra $(\{0, \frac{1}{2}, 1\}, \vee, \wedge, \oplus, \otimes, \neg, 0, \frac{1}{2}, 1)$ is functionally equivalent to the 3-element Post algebra P_3. Indeed, it is enough to express the cyclic negation $\sim x = (\frac{1}{2} \otimes x) \vee (\neg x \otimes \neg x)$.

MV-algebras are the algebraic counterpart of the infinite valued Lukasiewicz sentential calculus, as Boolean algebras are concerning the classical propositional logic. In contrast with what happens for Boolean algebras, some MV-algebras are not semi-simple, i.e. the intersection of their maximal ideals (the radical of A) is different from $\{0\}$. The simple example of non semi-simple MV-algebra is given by C. Chang in [1] (the algebra C). The MV-algebras generated by their radical are called perfect.

Mundici [4] defined correspondence functor Γ between MV-algebras and lattice-ordered abelian groups (abelian l-groups) with strong unit, and proved that Γ is a categorical equivalence. We define analogical functor Γ_c of P_3^ω- algebras and l-groups with strong unit u. More precisely, for every abelian l-group G, the functor Γ_c equips the unit interval $[0, 2u]$ with the operations: $x \vee y = max(x, y), x \wedge y = min(x, y), x \oplus y = 2u \wedge (x + y), x \otimes y = 0 \vee (x + y - 2u), \neg x = 2u - x, 1 = 2u$.

Notations.

(i) $D_0 = \Gamma(Z, 2) \cong P_3$, with 1 as a strong unit.

(ii) $D_1 = D = \Gamma_c(Z \times_{lex} Z, (2, 0))$ with the strong unit $(1, 0)$, the generator $d_1 (= (0, 1))$, and $\times_{lex}$ is the lexicographic product.

(iii) $D_m = \Gamma_c(Z \times_{lex} ... \times_{lex} Z, (2, 0, ..., 0))$ with the strong unit $(1, 0, ..., 0)$, the generators $d_1 (= (0, 0, ..., 1)), ..., d_m (= (0, 1, ..., 0))$, where the number of factors of Z is equal to $m + 1$.

(iv) Let D^*_m be the subalgebra of D_m generated by the radical (intersection of all maximal ideals) of D_m, where $m \in Z^+$.

Proposition: Let G be an abelian l-group with the strong unit u. Then $\Gamma_c(G, 2u)$ is a generalized P^{ω}_3-algebra $([0, 2u], \vee, \wedge, \oplus, \otimes, \neg, 0, u, 2u)$.

A subset F of a P^{ω}_3-algebra A is said to be an *ideal* if 1) $0 \in I$, 2) if $x, y \in I$, then $x \oplus y \in I$, and 3) if $x \in I$ and $y \leq x$, then $y \in I$.

Theorem:

1) D generates the variety $\mathbf{P}^{\omega}_3$.

2) There exists lattice isomorphism between the lattice of ideals of a P^{ω}_3-algebra A and the lattice of congruences of a P^{ω}_3-algebra A.

3) m-generated free P^{ω}_3-algebra is isomorphic to $D^{*^{3^m}}_m$.

5) The P^{ω}_3-algebras P_3 and D^m are projective for every $m \in Z^+$.

6) The variety $\mathbf{P}_3$ of 3-valued Post algebras is the only proper subvariety of the variety $\mathbf{P}^{\omega}_3$.

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