On (modal) expansions of pointed Abelian logic

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Abstract

The variety of lattice-ordered Abelian groups (Abelian ℓ-groups, for short) is well known and studied [5]. It was established in [7] that, as a quasivariety, it is generated by the ℓ-group of integer numbers. Abelian ℓ-groups are not only interesting from the algebraic point of view but also from the logical point of view, since they form the algebraic semantics for Abelian logic (see [1, 8]). Since the quasivariety of Abelian ℓ-groups have no proper subquasivarieties, Abelian logic has no proper extensions by finitary rules.

Another logical motivation for the study of Abelian ℓ-groups is the key role they play in understanding Łukasiewicz logic; see for example the proof of Chang's theorem [2]. In general, the study of Abelian ℓ-groups is very closely related to the study of MV-algebras, since these two classes of structures are connected via Mundici functor [2]. It is also worth mentioning that Abelian logic can be seen in as a weakening-free variant of Łukasiewicz logic.

In this talk, we will discuss the class of pointed Abelian ℓ-groups and its corresponding logic, which we call pointed Abelian logic. By a pointed Abelian ℓ-group we mean an Abelian ℓ-group with one additional fixed element in the signature without any additional property. While it may seem that it is not more than a cosmetic change, it turns out that the additional constant symbol allows us to express many new no-trivial logical axiom/rules and so the corresponding lattice of sub(quasi)varieties of pointed Abelian ℓ-groups is quite complex and worth exploring.

In previous research we have discussed several extensions of pointed Abelian logic, the most important of which was a finitary version of the unbounded Łukasiewicz logic. This logic was introduced (but not named) in [3], along with its philosophical and linguistic motivation, and has a clear mathematical motivation: it combines Abelian logic and Łukasiewicz logic in a very natural way.

In this talk, we focus on an infinitary version of the unbounded Łukasiewicz logic, i.e., a logic strongly complete with respect to the pointed ℓ-group of reals with point at -1.

To this end, we must first prove some general results about extensions of (pointed) Abelian logic using infinitary rules (recall here that we cannot get any non-trivial extension of Abelian logic by using additional finitary rules). We will give axiomatizations of some of these extensions and show that there are uncountably many of them. In particular, we focus on those whose corresponding algebraic semantics is the generalized subquasivariety generated by Archimedean ℓ-groups, integers and reals (in pointed case with positive or negative interpretation of the point).

We conclude the talk by exploring the addition of modalities to (the extensions of) pointed Abelian logic. We follow the footsteps of [4] (for Abelian logic) and [4, 6] (for Łukasiewicz logic) and focus on the differences caused by the presence of the additional constant and the lack of weakening.

References

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