Pretabular Tense Logics over $\mathsf{S4}_t$

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A logic L is called tabular if $L = \mathsf{Log}(\mathfrak{A})$ for some finite algebra $\mathfrak{A}$. L is called pretabular if L itself is not tabular while all of its proper consistent extensions are tabular. Let $\mathsf{Pre}(L)$ denote the set of pretabular logics extending L. It is proved in [7] that $|\mathsf{Pre}(\mathsf{Int})| = 3$. It was shown in [8, 4] that $|\mathsf{Pre}(\mathsf{S4})| = 5$. Moreover, [1] proved that $|\mathsf{Pre}(\mathsf{K4})| = 2^{\aleph_0}$. However, the tense case is more involved and we know much less about it. [6] introduced a pretabular tense logic $\mathsf{Ga} \in \mathsf{NExt}(\mathsf{S4}_t)$, whose frames have a maximum depth and width of 2 and do not contain any proper clusters.[1] It is claimed in [10] that $|\mathsf{Pre}(\mathsf{S4}_t)| \geq \aleph_0$ without a proof.

In this work, we study pretabular tense logics in the lattice $\mathsf{NExt}(\mathsf{S4}_t)$. We start with the sublattice $\mathsf{NExt}(\mathsf{S4.3}_t)$, where $\mathsf{S4.3}_t = \mathsf{S4} \oplus \{\boxtimes(\boxtimes p \to q) \vee \boxtimes(\boxtimes q \to p) : \boxtimes \in \{\Box, \blacksquare\}\}$ is the tense logic of chains. It turns out that the lattice $\mathsf{NExt}(\mathsf{S4.3}_t)$ is already much more complex than the lattice $\mathsf{NExt}(\mathsf{S4.3})$. It was shown in [5, 2] that every modal logic in $\mathsf{NExt}(\mathsf{S4.3})$ is finitely axiomatizable and enjoys the finite model property. However, $\mathsf{NExt}(\mathsf{S4.3}_t)$ contains infinitely many incomplete tense logics (see [11]). We obtain a full characterization of pretabular tense logics over $\mathsf{S4.3}_t$ as follows:

Theorem 1. *There are exactly five pretabular tense logics in* $\mathsf{NExt}(\mathsf{S4.3}_t)$. *More precisely,*

$$\mathsf{Pre}(\mathsf{S4.3}_t) = \{L_i : i < 5\}, \text{ where } L_i = \bigcap_{n\in\omega} \mathsf{Log}_t(\mathfrak{C}_i^n).\text{[2]}$$

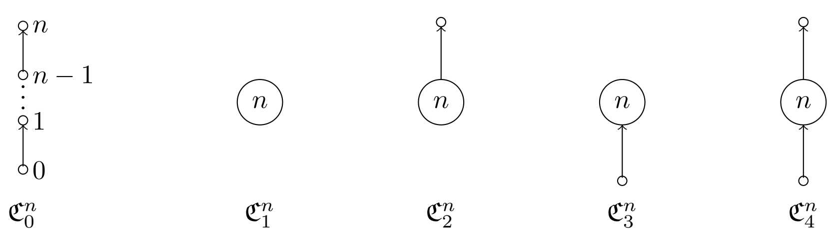


Figure 1: Frames $\mathfrak{C}_i^n$

It is clear that $\mathsf{Pre}(\mathsf{S4.3}) = \{\bigcap_{n\in\omega} \mathsf{Log}_\Diamond(\mathfrak{C}_i^n) : i < 3\}$, where $\mathsf{Log}_\Diamond(\mathfrak{C}_i^n)$ is the modal logic of $\mathfrak{C}_i^n$. The interaction between tense operators lead to new pretabular logics L_3 and L_4.

We generalize the results above and consider the lattices $\mathsf{NExt}(\mathsf{S4.3}_t^+)$ and $\mathsf{NExt}(\mathsf{S4.3}_t^-)$, where $\mathsf{S4.3}_t^+ = \mathsf{S4} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$ and $\mathsf{S4.3}_t^- = \mathsf{S4} \oplus \blacksquare(\blacksquare p \to q) \vee \blacksquare(\blacksquare q \to p)$. The bi-intuitionistic logic of 'co-trees' was studied in [9]. $\mathsf{S4.3}_t^+$ and $\mathsf{S4.3}_t^-$ are the tense logics of 'co-trees' and 'trees', respectively. The main result we have for them is as follows:

Theorem 2. $|\mathsf{Pre}(\mathsf{S4.3}_t^+)| = |\mathsf{Pre}(\mathsf{S4.3}_t^-)| = 12.$

Pretabular tense logics in $\mathsf{Pre}(\mathsf{S4.3}_t^-) \setminus \mathsf{Pre}(\mathsf{S4.3}_t)$ are characterized by the classes of finite frames given in Figure 2. In the modal case, only the forks generate a pretabular logic.

[1] $\mathsf{S4}_t$ is the tense logic of reflexive and transitive frames.

[2] $\mathfrak{C}_i^n$ are frames depicted in Figure 1. ⓝ in the figures denotes a cluster with n points.

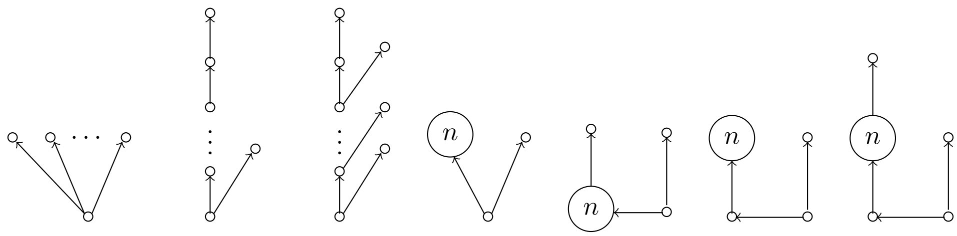
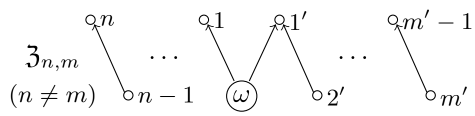


Figure 2: Frames for logics in $\mathsf{Pre}(\mathsf{S4.3}_t^-) \setminus \mathsf{Pre}(\mathsf{S4.3}_t)$

Once we allow 'zigzag-like' frames, we are in completely different situation, even if we put strong constraints on the depth and width of the frames. We consider the tense logic $\mathsf{BS}^2_{2,2} = \mathsf{S4}_t \oplus \{\mathsf{bd}_2, \mathsf{bw}_2^+, \mathsf{bw}_2^-\}$, where $\mathsf{bd}_2, \mathsf{bw}_2^+$ and bw_2^- are defined as in [3]. $\mathsf{BS}^2_{2,2}$ is exactly the tense logic of 'zigzags' with clusters. We obtain also a full characterization of pretabular tense logics in $\mathsf{NExt}(\mathsf{BS}^2_{2,2})$ as follows:

Theorem 3. *Let $L \in \mathsf{NExt}(\mathsf{BS}^2_{2,2})$. Then L is pretabular if and only if $L = \mathsf{Ga}$ or $L = \mathsf{Log}(\mathfrak{F})$ for some $\mathfrak{F} \in \mathcal{Z} \cup \breve{\mathcal{Z}}$, where $\mathcal{Z}$ is the class of frames depicted in the figure below.*[3]



Corollary 4. $|\mathsf{Pre}(\mathsf{BS}^2_{2,2})| = \aleph_0$.

We construct infinitely many pretabular tense logics in $\mathsf{NExt}(\mathsf{S4}_t)$, which provides a proof for the claim in [10]. The next step is to investigate the set $\mathsf{Pre}(\mathsf{S4}_t)$ of pretabular logics and our conjecture is that $|\mathsf{Pre}(\mathsf{S4}_t)| = 2^{\aleph_0}$.

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[3] $\breve{\mathcal{Z}} = \{(W, R^{-1}) : (W, R) \in \mathcal{Z}\}$ and $\textcircled{\omega}$ in the figure denotes a cluster with ω points.