Effective Descent Morphisms in the Dual Categories of ((Compact) Hausdorff) Topological Spaces, Banach spaces, and Some Other Concrete Categories

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Descent theory begins with Grothendieck's remarkable work [4]. The context it uses consists of a fibration of categories over a category $\mathcal{C}$, and a morphism $p : E \to B$ in $\mathcal{C}$; the morphism p is called an effective descent morphism (resp. descent morphism) if the canonical morphism from the fibre over B to a certain category of descent data that is described in terms of the fibre over E is an equivalence of categories (resp. is full and faithful). Usually one restricts, however, to the case of basic fibration, where the fibres over B and E are comma categories $\mathcal{C}/B$ and $\mathcal{C}/E$, respectively. Later, independently J. Beck (unpublished), and J. Bénabou and J. Roubaud [1] worked out the monadic approach to descent theory. In particular, it was proved that in the case of basic fibration, a morphism p is an effective descent morphism (resp. descent morphism) if and only if the pullback functor $p^* : \mathcal{C}/B \to \mathcal{C}/E$ is monadic (resp. premonadic). In many concrete situations, describing such morphisms is a highly non-trivial problem, with many publications of various authors devoted to it. For instance, effective descent morphisms in the category of topological spaces were characterized by Reiterman and Tholen [6], Clementino and Hofmann [2], Clementino and Janelidze [3]. Effective descent morphisms in the category of Hausdorff spaces were described by Clementino and Janelidze [3].

In [10], we reduced the problem whether all descent morphisms are effective in a category with a factorization system $(\mathbb{E}, \mathbb{M})$ (with $\mathbb{M} \subseteq M$) to the simpler one. Namely, we have shown that this problem is equivalent to the one obtained from it by replacing "all descent morphisms" by "all descent morphisms from $\mathbb{E}$", and by replacing arbitrary descent data in the definition of an effective descent morphism by descent data of a certain kind. The goal of this talk is to present some applications of this simplification. Below we use "codescent" for descent in dual categories.

A new proof of the following fact is found: *every regular monomorphism in the category of topological spaces (i.e., an embedding) is an effective codescent morphism.* Note that, initially, this fact was proved by Mantovani in a different way (unpublished). The third proof of this fact arises from the results on effective descent morphisms in topological categories given in [10]. The similar results are obtained for the categories of uniform spaces, proximity spaces, and some other topological categories.

The problem when a functor reflects effective descent morphisms is simplified: as different from the similar results known earlier, we require a functor to preserve not all pullbacks, but pullbacks of $\mathbb{E}$-morphisms. This enables us to study effective codescent morphisms in the duals of several more categories of topological nature since, in topology, there is a number of forgetful functors which do not preserve pushouts, but preserve pushouts of some monomorphisms. In particular, the following statement is obtained: *every codescent morphism in the category of Hausdorff spaces is effective.*

Note that not every regular monomorphism in the category of Hausdorff spaces (i.e. a closed embedding) is a codescent morphism (Kelly [5]). We gave the following statement. Before we formulate it, recall that subsets U_1 and U_2 of a topological space B are called completely separable if there exists a continuous mapping from B to the closed interval $[0, 1]$ such that $f(U_1) = \{0\}$ and $f(U_2) = \{1\}$.

For a closed embedding $p : B \rightarrowtail E$ *in the category of Hausdorff spaces and, for the following conditions, one has (i)⇒(ii)⇐(iii). If* E *is regular and* B *is compact, then (i) and (ii) are equivalent. If, again,* E *is regular and each two disjoint open subsets of* B *are completely separable, then all three conditions are equivalent: (i)* p *is an effective codescent morphism; (ii) for any completely separable open subsets* U_1 *and* U_2 *of* B*, there exist disjoint open subsets* V_1 *and* V_2 *of* E *such that* $U_1 = B \cap V_1$ *and* $U_2 = B \cap V_2$*; (iii) for any disjoint open subsets* U_1 *and* U_2 *of* B*, there exist disjoint open subsets* V_1 *and* V_2 *of* E *such that* $U_1 = B \cap V_1$ *and* $U_2 = B \cap V_2$*.*

Further, the following statements are obtained: *every monomorphism in the category of compact Hausdorff spaces (i.e., an injective continuous mapping) is an effective codescent morphism. Every regular monomorphism (i.e., an isometric embedding) in the category of Banach spaces (with linear contractions) is an effective codescent morphism.*

Effective descent morphisms in topological categories are studied. The obtained results imply that *if* $\mathcal{V}$ *be a Mal'cev variety of universal algebras, then every regular epimorphism (i.e., a continuous open surjective homomorphism) is an effective descent morphism in the category of topological* $\mathcal{V}$*-algebras.*

Finally, note that, with the aid of the above-mentioned simplification of the descent problem, effective codescent morphisms are characterized in some varieties of universal algebras [9], [11], [7], [8], [12].

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