Extending the Blok-Esakia Theorem to the monadic setting

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It is a classic result of McKinsey and Tarski [6] that the Gödel translation embeds the intuitionistic propositional calculus IPC into the propositional modal logic $\mathsf{S4}$. The normal extensions of $\mathsf{S4}$ into which the Gödel translation embeds a given superintuitionistic logic L are called the *modal companions* of L. Esakia's Theorem [3] states that the largest modal companion of IPC is the Grzegorczyk logic $\mathsf{Grz} \coloneqq \mathsf{S4} + \mathsf{grz}$, where

$$\mathsf{grz} = \Box(\Box(p \to \Box p) \to p) \to p.$$

Every superintuitionistic logic L has a least and a greatest modal companion, denoted $\tau\mathsf{L}$ and $\sigma\mathsf{L}$, and hence the modal companions of L form an interval $[\tau\mathsf{L}, \sigma\mathsf{L}]$ in the lattice of normal extensions of $\mathsf{S4}$. The celebrated Blok-Esakia Theorem, established independently by Blok [2] and Esakia [3], states that mapping L to $\sigma\mathsf{L}$ yields an isomorphism between the lattice of superintuitionistic logics and the lattice of normal extensions of Grz.

The predicate extension of the Gödel translation embeds the intuitionistic predicate calculus IQC into the predicate $\mathsf{S4}$ logic $\mathsf{QS4}$. However, the behavior of modal companions of predicate superintuitionistic logics is much less understood. It is convenient to first investigate the restriction of the predicate Gödel translation to the *monadic fragments* (also known as the *one-variable fragments*) MIPC of IQC and $\mathsf{MS4}$ of $\mathsf{QS4}$, which can be thought of as bimodal logics and can be studied using the standard semantic tools of modal logic. Fischer Servi [5] proved that the Gödel translation embeds MIPC into $\mathsf{MS4}$ and Esakia [4] showed that the monadic Grzegorczyk logic $\mathsf{MGrz} \coloneqq \mathsf{MS4} + \mathsf{grz}$ is a modal companion of MIPC. It is then natural to wonder whether an analogue of the Blok-Esakia Theorem holds in the monadic setting.

This talk will address the challenges involved in the study of modal companions of extensions of MIPC. We will see that the Blok-Esakia Theorem fails in the monadic setting: the map from the lattice of extensions of MIPC to the lattice of extensions of MGrz that naturally generalizes σ is not a lattice isomorphism. We will then discuss the obstacles to the generalization of Esakia's Theorem to MIPC. Some possible ways to recover positive results in the monadic setting will also be mentioned. This talk is based on joint work with G. Bezhanishvili and most of the discussed results can be found in [1].

References

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