Projectivity in quasivarieties of logic

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This talk is about some *bridge theorems*, i.e. connections between syntactic features of a logic and the properties of its algebraic semantics, and the role that projective algebras have in them. In particular, we will be concerned with algebraizable logics in the sense of Blok-Pigozzi [3], whose equivalent algebraic semantics are quasivarieties of algebras.

Given a quasivariety Q, an algebra $\mathbf{P}$ is *projective* in Q if and only if it is a retract of a free algebra in Q. Thus, projective algebras extend the class of free algebras, and they can "play the role" of free algebras in some instances as we will discuss. It is important to note that projectivity is a categorical notion, and as such is preserved by categorical equivalence, while free algebras in general are not. This allows the study of projective algebras via categorical equivalences or dualities with respect to objects that are easier to understand.

In the first part of this talk we will present some examples of characterizations of projective algebras in relevant (quasi)varieties related to logic, involving techniques that vary from purely algebraic to duality theoretic. We will present some examples from the literature, and some recent results from [1]. The examples mostly lie in the framework of residuated lattices, which provide the equivalent algebraic semantics of substructural logics; the latter comprise classical logic and many of the most interesting nonclassical logics, e.g. intuitionistic logic, relevance logics, linear logic, many-valued logics.

In the second part of the talk, we discuss the role of projective algebras in the algebraic study of unification problems, in the setting developed by Ghilardi [4], together with applications to the study of the structural completeness of a quasivariety and its weakenings [2].

Finally we present a new approach, based on recent joint work with Tommaso Flaminio, to *equational anti-unification problems*, whose synctactic version was first introduced in the 1970s to study inductive proofs. The key is once again the use of projective algebras. We show that both equational anti-unification problems and their type (i.e., the cardinality of the set of "best" - or least general - solutions) can be studied algebraically; we discuss some relevant examples, e.g., in Boolean algebras, Kleene algebras, Gödel algebras, MV-algebras (the equivalent algebraic semantics of, respectively, classical logic, 3-valued Kleene logic, Gödel-Dummett logic, infinite-valued Łukasiewicz logic).

References

[1] P. Aglianò and S. Ugolini, *Projectivity in (bounded) commutative integral residuated lattices*, Algebra Universalis **84** (2), 2023.

[2] P. Aglianò and S. Ugolini, *Structural and universal completeness in algebra and logic*, Annals of Pure and Applied Logic **175**(3), 2024.

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[4] S. Ghilardi, *Unification through projectivity*, Journal of Logic and Computation **7**(6), 733–752, 1997.