The mathematical theory of contextuality
Lecture 2: sheaf-thereoretic formulation

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In terms of measures:

$$\mathsf{D}(f)(d)(S) \;=\; d(f^{-1}(S)).$$

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Normalization corresponds to this monad being **affine**

$$\mathsf{D}(\mathbf{1}) \;\cong\; \mathbf{1}.$$

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Features in **tropical geometry** (the **max-plus** semiring).

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This yields a functor $\mathcal{D}_R : \mathbf{Set} \longrightarrow \mathbf{Set}$.

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$$\rho_U^V : P(V) \to P(U)$$

subject to the functoriality requirements: if $U \subseteq V \subseteq W$, then

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Functoriality is easily verified: in this notation

$$(f|_V)_U = f|_U.$$

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- However, there is an important conceptual aspect which should be understood. Presheaves allow us to formalise the concept of **variable set**. The variation is essentially over **contexts**. So presheaves provide the natural setting for talking about contextuality!

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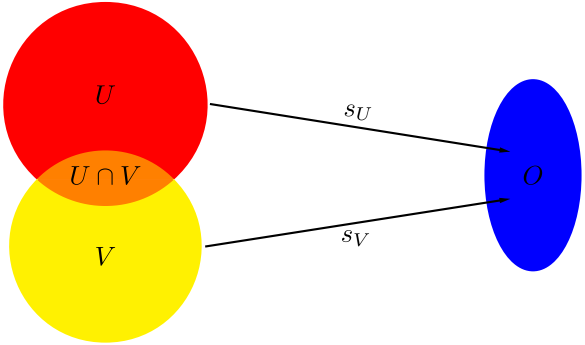
The presheaf P is a **sheaf** if for every open cover $\mathcal{U}$, it satisfies the sheaf condition for $\mathcal{U}$.

Gluing functional sections



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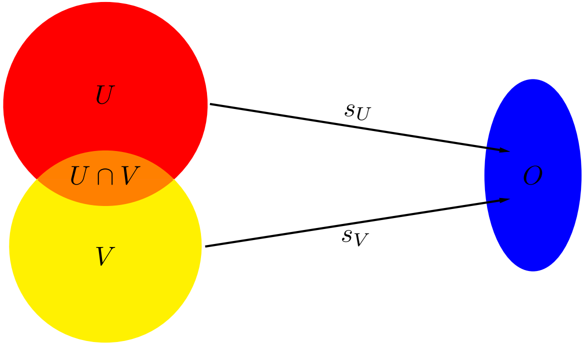

If $s_U|_{U \cap V} = s_V|_{U \cap V}$, they can be glued to form

$$s : U \cup V \longrightarrow O$$

such that $s|_U = s_U$ and $s|_V = s_V$.

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In particular, this is one of the main intuitions behind **sheaf cohomology**.

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This presheaf is easily seen to satisfy the sheaf condition.

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- Set of variables X (we will think of this as a discrete space)
- Set of outcomes O

The presheaf of events:

$$\mathcal{E} : \mathcal{P}(X)^{\mathsf{op}} \to \mathbf{Set} \quad :: \quad U \mapsto O^U$$

Conceptually, a local section $s \in \mathcal{E}(U)$ represents the event of measuring or observing the variables $x \in U$, and observing the outcomes or values $s(x)$.

Restriction is by function restriction:

$$\rho^U_V(s) = s|_V, \qquad V \subseteq U$$

This presheaf is easily seen to satisfy the sheaf condition.

A useful generalization: we have a set O_x of outcomes for each measurement x. Then $\mathcal{E}(U) = \prod_{x \in U} O_x$. Restriction is by projection.

A Deterministic Universe is Classical

As we shall see, contextuality arises exactly where the sheaf property **fails**. Contextuality witnesses – Bell tests and other forms we will study – are **exactly** witnesses to this failure – **obstructions** to gluing.

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This is in fact the content of the Conway-Kochen "Free Will Theorem".

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Rather than a fixed probability space (X, d), $d \in \mathcal{D}_R(X)$, we can now consider a **variable probability space**

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We shall now see how this arises naturally in some important situations.

A Probabilistic Model Of An Experiment

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Example: The Bell Model

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
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a_2	b_1	3/8	1/8	1/8	3/8
a_2	b_2	1/8	3/8	3/8	1/8

The entry in row 2 column 3 says:

If Alice looks at a_1 and Bob looks at b_2, then 1/8th of the time, Alice sees a 0 and Bob sees a 1.

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$$\{a, b\}, \quad \{a', b\}, \quad \{a, b'\}, \quad \{a', b'\}.$$

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Each row of the table specifies a **probability distribution** on events O^C for a given choice of measurements C.

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The different sets of compatible measurements correspond to the different contexts of measurement and observation of the physical system.

The fact that the behaviour of these observable outcomes cannot be accounted for by some context-independent global description of reality corresponds to the geometric fact that these local sections cannot be glued together into a **global section**.

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The quantum phenomena of **non-locality** and **contextuality** correspond exactly to the existence of obstructions to global sections in this sense.

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E.g. in the bipartite case, consider $C = \{m_a, m_b\}$, $C' = \{m_a, m'_b\}$.
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This says that the probability for Alice to get the outcome $s_0(m_a)$ is the same, whether we marginalize over the possible outcomes for Bob with measurement m_b, or with m'_b.

In other words, Bob's choice of measurement cannot influence Alice's outcome.

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If d is a global section for the model $\{e_C\}$, we recover the predictions of the model by **averaging over the values of these hidden variables**:

$$e_C(s) \;=\; d|C(s) \;=\; \sum_{s' \in \mathcal{E}(X), s'|C = s} d(s') \;=\; \sum_{s' \in \mathcal{E}(X)} \delta_{s'|C}(s) \cdot d(s').$$

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- O is set of outcomes or values for the variables, which we take to be the same in each fibre.

We have a sheaf of sets over $\mathcal{P}(X)$, namely $\mathcal{E} :: U \longmapsto O^U$ with restriction

$$\mathcal{E}(U \subseteq U') \colon \mathcal{E}(U') \longrightarrow \mathcal{E}(U) :: s \longmapsto s|U \ .$$

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A probability table can be represented by a family $\{p_C\}_{C \in \mathcal{M}}$ with p_C a probability distribution on $\mathcal{E}(C) = O^C$, where contexts C corresponds to the rows of the table.

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The logical and strong forms of contextuality are concerned with **possibilities**, which can be represented by a subpresheaf $\mathcal{S}$ of $\mathcal{E}$, where for each context $U \subseteq X$, $\mathcal{S}(U) \subseteq O^U$ is the set of all possible outcomes.

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Explicitly, $\mathcal{S}$ is defined as follows, where $\mathsf{supp}(p_C|U \cap C)$ is the support of the marginal of p_C at $U \cap C$.

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We can use this formalisation to characterize contextuality as follows.

Definition

For any empirical model $\mathcal{S}$:

- For all $C \in \mathcal{M}$ and $s \in \mathcal{S}(C)$, $\mathcal{S}$ is **logically contextual** at s, written $\mathsf{LC}(\mathcal{S}, s)$, if s is not a member of any compatible family.
- $\mathcal{S}$ is **strongly contextual**, written $\mathsf{SC}(\mathcal{S})$, if $\mathsf{LC}(\mathcal{S}, s)$ for all s. Equivalently, if it has no global section, *i.e.* if $\mathcal{S}(X) = \varnothing$.

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There is also a natural projection function

$$p : X \to I \qquad p : (i, x) \mapsto i$$

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There is also a natural projection function

$$p : X \to I \qquad p : (i, x) \mapsto i$$

Conversely, given $p : X \to I$, we can form the indexed family $\{X_i\}_{i \in I}$, where $X_i := p^{-1}(\{i\})$.

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Sheaves on X are equivalently formulated as continuous maps $p : Y \to X$ which are **local homeomorphisms** (*espaces étalé*).

Bundle Pictures

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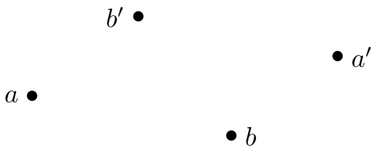

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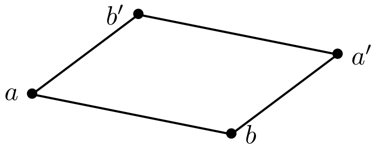

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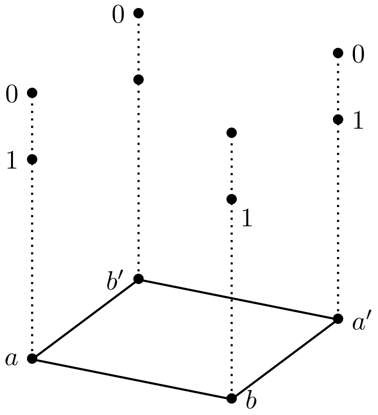

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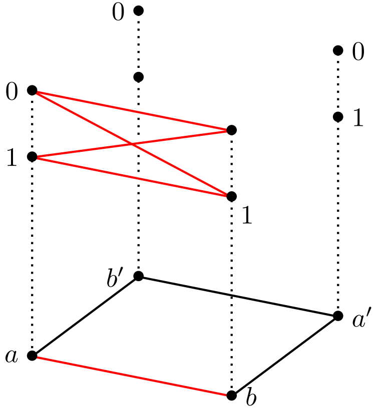

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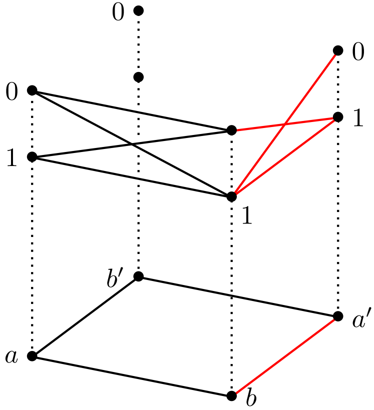

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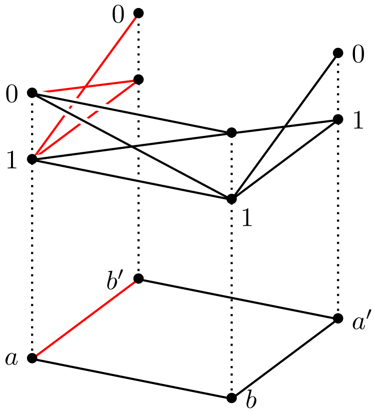

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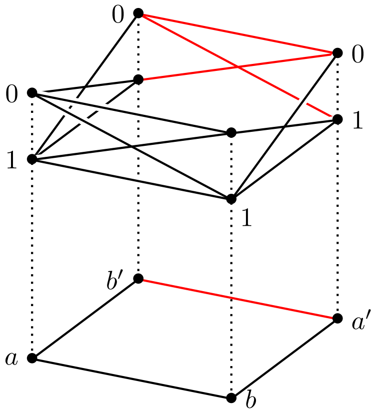

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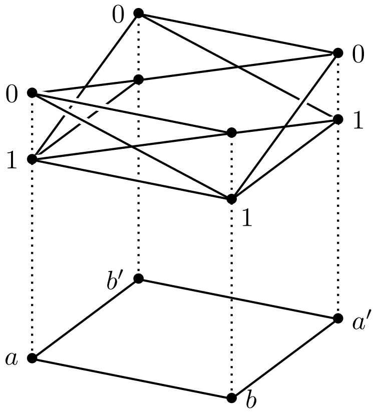

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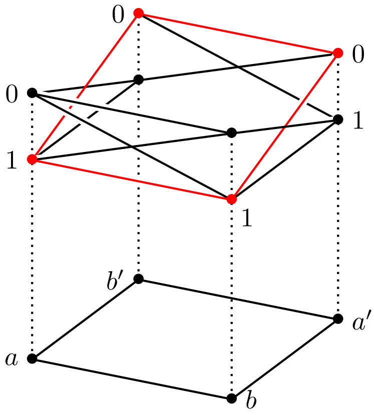

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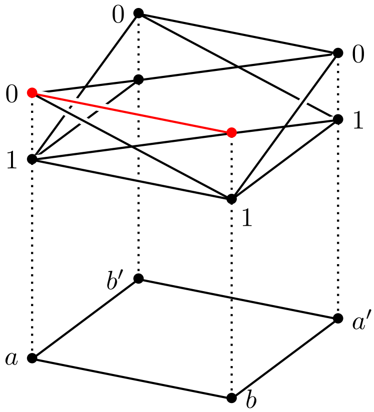

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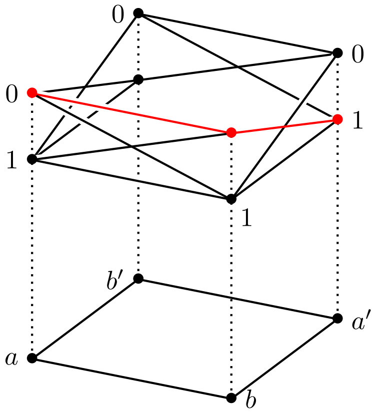

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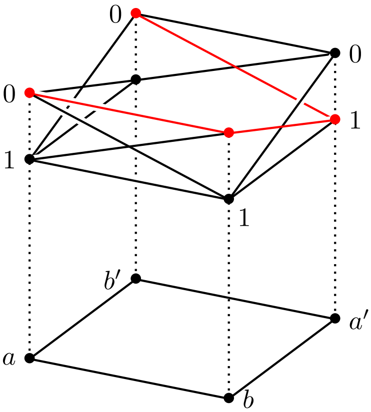

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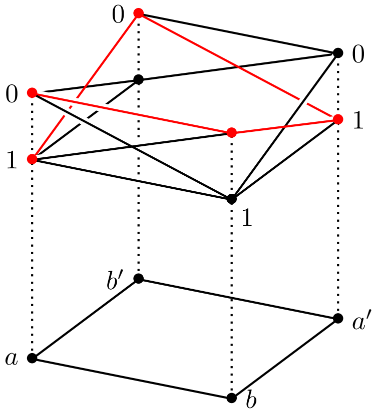

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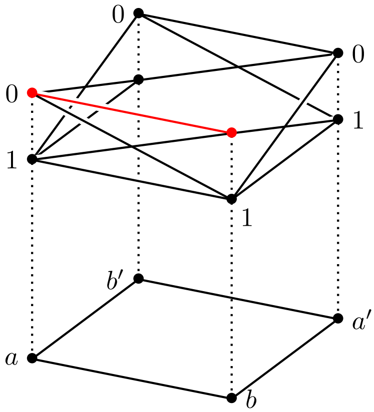

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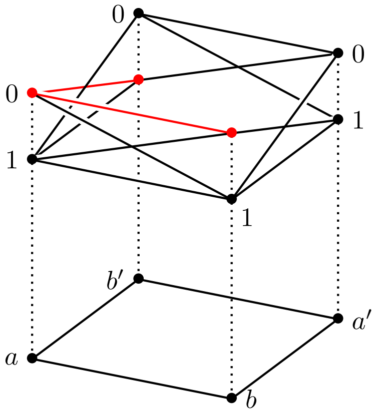

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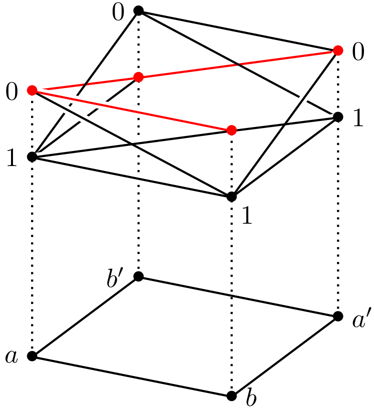

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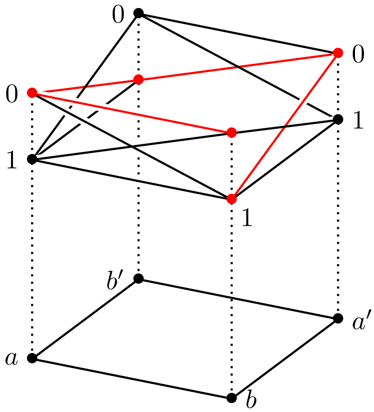

Strong Contextuality

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

Bundle Pictures

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ab'	✓	×	×	✓
$a'b$	✓	×	×	✓
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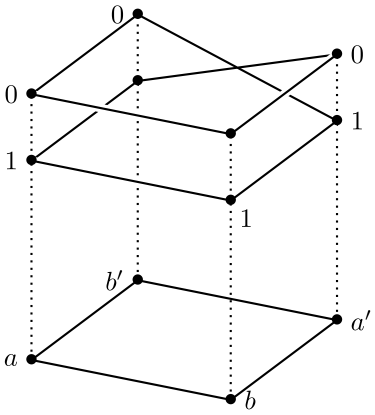
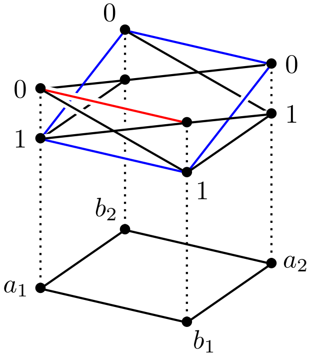
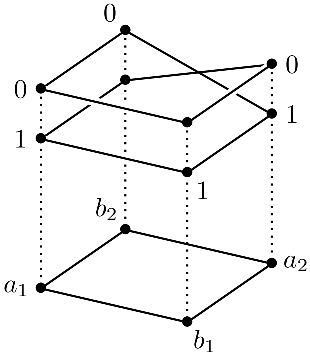


Visualizing Contextuality





The Hardy table and the PR box as bundles

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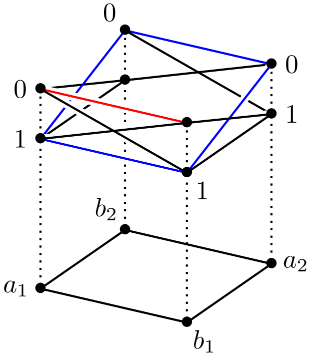

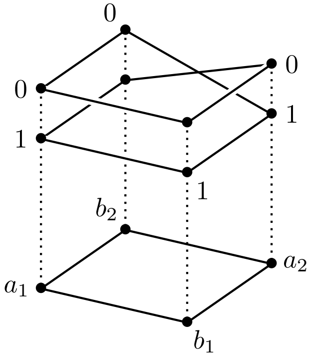

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A hierarchy of degrees of contextuality:

$$\text{Bell} \;<\; \text{Hardy} \;<\; \text{GHZ}$$

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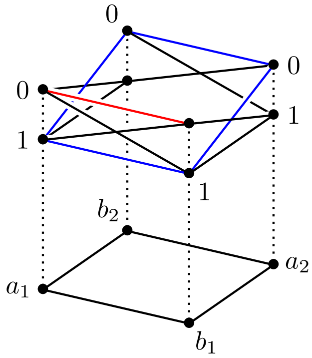

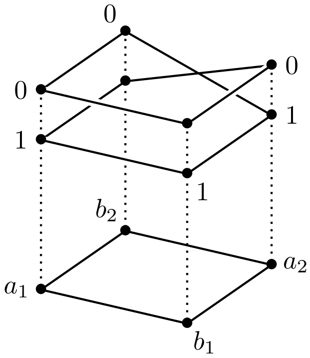

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  Obviously, strong non-locality implies logical non-locality.

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