THE MATHEMATICAL THEORY OF CONTEXTUALITY
Lecture 4: Partial Boolean Algebras

Samson Abramsky

s.abramsky@ucl.ac.uk

Rui Soares Barbosa

rui.soaresbarbosa@inl.int

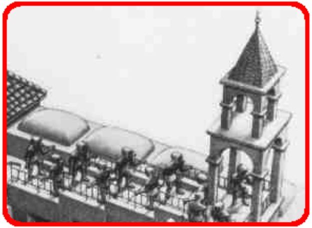
TACL 2024 Summer School

The essence of contextuality

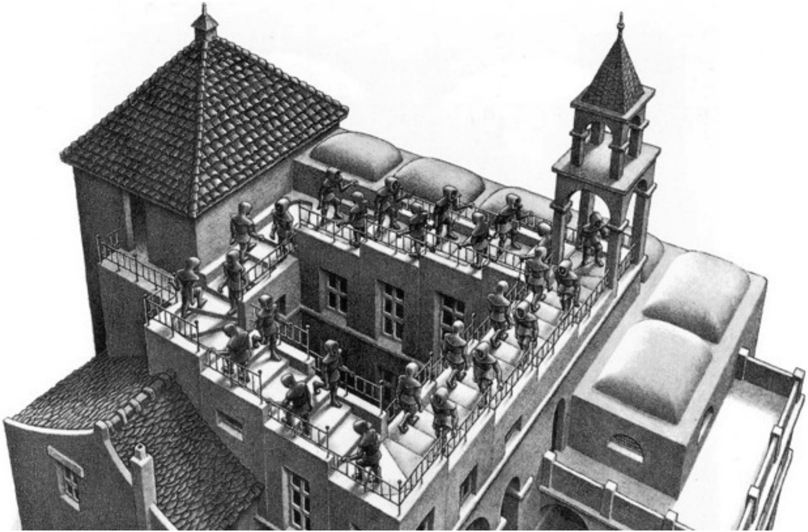
- Not all properties may be observed simultaneously.
- Sets of jointly observable properties provide **partial, classical snapshots**.
- Contextuality arises where there is a family of data which is

locally consistent but globally inconsistent

Contextuality Analogy: Local Consistency



Contextuality Analogy: Global Inconsistency



Brief review of Hilbert spaces

Brief review of Hilbert spaces

Hilbert space is a complex inner product space. There is a norm defined from the inner product, and the space has to be complete in this norm.

Brief review of Hilbert spaces

Hilbert space is a complex inner product space. There is a norm defined from the inner product, and the space has to be complete in this norm.

The salient notion of basis is **orthonormal basis**: a basis consisting of pairwise orthogonal unit vectors.

Brief review of Hilbert spaces

Hilbert space is a complex inner product space. There is a norm defined from the inner product, and the space has to be complete in this norm.

The salient notion of basis is **orthonormal basis**: a basis consisting of pairwise orthogonal unit vectors.

Up to isomorphism, **there is only one Hilbert space in each dimension**.

Brief review of Hilbert spaces

Hilbert space is a complex inner product space. There is a norm defined from the inner product, and the space has to be complete in this norm.

The salient notion of basis is **orthonormal basis**: a basis consisting of pairwise orthogonal unit vectors.

Up to isomorphism, **there is only one Hilbert space in each dimension**.

So for ordinary QM, the possibilities are (in principle) just $\mathbb{C}^n$ and $\ell_2(\omega)$.

Brief review of Hilbert spaces

Hilbert space is a complex inner product space. There is a norm defined from the inner product, and the space has to be complete in this norm.

The salient notion of basis is **orthonormal basis**: a basis consisting of pairwise orthogonal unit vectors.

Up to isomorphism, **there is only one Hilbert space in each dimension**.

So for ordinary QM, the possibilities are (in principle) just $\mathbb{C}^n$ and $\ell_2(\omega)$.

C^* algebras are an elegant algebraic approach, but not really more general: by the Gelfand-Naimark theorem, every C^* algebra is isomorphic to a subalgebra of $B(\mathcal{H})$.

Brief review of Hilbert spaces

Hilbert space is a complex inner product space. There is a norm defined from the inner product, and the space has to be complete in this norm.

The salient notion of basis is **orthonormal basis**: a basis consisting of pairwise orthogonal unit vectors.

Up to isomorphism, **there is only one Hilbert space in each dimension**.

So for ordinary QM, the possibilities are (in principle) just $\mathbb{C}^n$ and $\ell_2(\omega)$.

C^* algebras are an elegant algebraic approach, but not really more general: by the Gelfand-Naimark theorem, every C^* algebra is isomorphic to a subalgebra of $B(\mathcal{H})$.

Quantum information mostly restricts consideration to finite dimensions: $\mathbb{C}^n$.

Brief review of Hilbert spaces

Hilbert space is a complex inner product space. There is a norm defined from the inner product, and the space has to be complete in this norm.

The salient notion of basis is **orthonormal basis**: a basis consisting of pairwise orthogonal unit vectors.

Up to isomorphism, **there is only one Hilbert space in each dimension**.

So for ordinary QM, the possibilities are (in principle) just $\mathbb{C}^n$ and $\ell_2(\omega)$.

C^* algebras are an elegant algebraic approach, but not really more general: by the Gelfand-Naimark theorem, every C^* algebra is isomorphic to a subalgebra of $B(\mathcal{H})$.

Quantum information mostly restricts consideration to finite dimensions: $\mathbb{C}^n$.

Finite dimensional linear algebra: isn't that trivial?

Brief review of Hilbert spaces

Hilbert space is a complex inner product space. There is a norm defined from the inner product, and the space has to be complete in this norm.

The salient notion of basis is **orthonormal basis**: a basis consisting of pairwise orthogonal unit vectors.

Up to isomorphism, **there is only one Hilbert space in each dimension**.

So for ordinary QM, the possibilities are (in principle) just $\mathbb{C}^n$ and $\ell_2(\omega)$.

C^* algebras are an elegant algebraic approach, but not really more general: by the Gelfand-Naimark theorem, every C^* algebra is isomorphic to a subalgebra of $B(\mathcal{H})$.

Quantum information mostly restricts consideration to finite dimensions: $\mathbb{C}^n$.

Finite dimensional linear algebra: isn't that trivial?

$\boxed{\textbf{No!}}$

Complex Matrices, bras and kets

Since we are working in finite dimensions, operators can be represented by complex matrices.

Complex Matrices, bras and kets

Since we are working in finite dimensions, operators can be represented by complex matrices.

Matrix transpose is A^T. The **adjoint** A^* is the conjugate transpose of A. Thus $[a_{i,j}]^* = [\overline{a_{j,i}}]$.

Complex Matrices, bras and kets

Since we are working in finite dimensions, operators can be represented by complex matrices.

Matrix transpose is A^T. The **adjoint** A^* is the conjugate transpose of A. Thus $[a_{i,j}]^* = [\overline{a_{j,i}}]$.

A **projector** P is a self-adjoint idempotent ($P^* = P^2 = P$).

Complex Matrices, bras and kets

Since we are working in finite dimensions, operators can be represented by complex matrices.

Matrix transpose is A^T. The **adjoint** A^* is the conjugate transpose of A. Thus $[a_{i,j}]^* = [\overline{a_{j,i}}]$.

A **projector** P is a self-adjoint idempotent ($P^* = P^2 = P$).

A self-adjoint A can be written (Spectral theorem) as $A = \sum_i \lambda_i P_i$, where the λ_i are real numbers (the eigenvalues), and $\sum_i P_i = I$.

Complex Matrices, bras and kets

Since we are working in finite dimensions, operators can be represented by complex matrices.

Matrix transpose is A^T. The **adjoint** A^* is the conjugate transpose of A. Thus $[a_{i,j}]^* = [\overline{a_{j,i}}]$.

A **projector** P is a self-adjoint idempotent ($P^* = P^2 = P$).

A self-adjoint A can be written (Spectral theorem) as $A = \sum_i \lambda_i P_i$, where the λ_i are real numbers (the eigenvalues), and $\sum_i P_i = I$.

A **ket** is a (column, $d \times 1$) vector. Thus for the qubit ($\mathbb{C}^2$), $|0\rangle = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $|1\rangle = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

Complex Matrices, bras and kets

Since we are working in finite dimensions, operators can be represented by complex matrices.

Matrix transpose is A^T. The **adjoint** A^* is the conjugate transpose of A. Thus $[a_{i,j}]^* = [\overline{a_{j,i}}]$.

A **projector** P is a self-adjoint idempotent ($P^* = P^2 = P$).

A self-adjoint A can be written (Spectral theorem) as $A = \sum_i \lambda_i P_i$, where the λ_i are real numbers (the eigenvalues), and $\sum_i P_i = I$.

A **ket** is a (column, $d \times 1$) vector. Thus for the qubit ($\mathbb{C}^2$), $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

A **bra** is the adjoint of a ket. We can multiply a bra ($1 \times d$) with a ket ($d \times 1$) to get a 1×1 matrix, which we identify with a scalar. This is just the complex inner product.

Complex Matrices, bras and kets

Since we are working in finite dimensions, operators can be represented by complex matrices.

Matrix transpose is A^T. The **adjoint** A^* is the conjugate transpose of A. Thus $[a_{i,j}]^* = [\overline{a_{j,i}}]$.

A **projector** P is a self-adjoint idempotent ($P^* = P^2 = P$).

A self-adjoint A can be written (Spectral theorem) as $A = \sum_i \lambda_i P_i$, where the λ_i are real numbers (the eigenvalues), and $\sum_i P_i = I$.

A **ket** is a (column, $d \times 1$) vector. Thus for the qubit ($\mathbb{C}^2$), $|0\rangle = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $|1\rangle = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

A **bra** is the adjoint of a ket. We can multiply a bra ($1 \times d$) with a ket ($d \times 1$) to get a 1×1 matrix, which we identify with a scalar. This is just the complex inner product.

If $A = [a_{i,j}]$ is a $m \times n$ matrix and B a $p \times q$ matrix, then the Kronecker product $A \otimes B := [a_{i,j}B]$ is an $mp \times nq$ matrix, which represents the tensor product of the corresponding linear maps.

Complex Matrices, bras and kets

Since we are working in finite dimensions, operators can be represented by complex matrices.

Matrix transpose is A^T. The **adjoint** A^* is the conjugate transpose of A. Thus $[a_{i,j}]^* = [\overline{a_{j,i}}]$.

A **projector** P is a self-adjoint idempotent ($P^* = P^2 = P$).

A self-adjoint A can be written (Spectral theorem) as $A = \sum_i \lambda_i P_i$, where the λ_i are real numbers (the eigenvalues), and $\sum_i P_i = I$.

A **ket** is a (column, $d \times 1$) vector. Thus for the qubit ($\mathbb{C}^2$), $|0\rangle = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $|1\rangle = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

A **bra** is the adjoint of a ket. We can multiply a bra ($1 \times d$) with a ket ($d \times 1$) to get a 1×1 matrix, which we identify with a scalar. This is just the complex inner product.

If $A = [a_{i,j}]$ is a $m \times n$ matrix and B a $p \times q$ matrix, then the Kronecker product $A \otimes B := [a_{i,j}B]$ is an $mp \times nq$ matrix, which represents the tensor product of the corresponding linear maps.

Categorically, the category of matrices is a monoidal (even compact closed) skeleton of the category of finite-dimensional Hilbert spaces.

Tensor Product

Tensor Product

Compound systems in QM are represented by tensor products $\mathcal{H} \otimes \mathcal{K}$ of the corresponding Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

Tensor Product

Compound systems in QM are represented by tensor products $\mathcal{H} \otimes \mathcal{K}$ of the corresponding Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

This is where Alice and Bob live!

Tensor Product

Compound systems in QM are represented by tensor products $\mathcal{H} \otimes \mathcal{K}$ of the corresponding Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

This is where Alice and Bob live!

If $\mathcal{H}$ has ONB $\{\psi_i\}$ and $\mathcal{K}$ has ONB $\{\phi_j\}$ then $\mathcal{H} \otimes \mathcal{K}$ has ONB $\{\psi_i \otimes \phi_j\}$.

Tensor Product

Compound systems in QM are represented by tensor products $\mathcal{H} \otimes \mathcal{K}$ of the corresponding Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

This is where Alice and Bob live!

If $\mathcal{H}$ has ONB $\{\psi_i\}$ and $\mathcal{K}$ has ONB $\{\phi_j\}$ then $\mathcal{H} \otimes \mathcal{K}$ has ONB $\{\psi_i \otimes \phi_j\}$.

If we represent qubit space with a standard basis $\{|0\rangle, |1\rangle\}$, then n-qubit space has basis

$$\{|s\rangle \mid s \in \{0,1\}^n\}$$

Projectors as quantum propositions

Quantum observables or (projective) measurements are defined by families of projectors $\{P_i\}_i$ with $\sum_i P_i = I$.

Projectors as quantum propositions

Quantum observables or (projective) measurements are defined by families of projectors $\{P_i\}_i$ with $\sum_i P_i = I$.

These **projective resolutions of the identity** give disjoint cases for the various possible outcomes i.

Projectors as quantum propositions

Quantum observables or (projective) measurements are defined by families of projectors $\{P_i\}_i$ with $\sum_i P_i = I$.

These **projective resolutions of the identity** give disjoint cases for the various possible outcomes i.

The commutator $[P, Q] := PQ - QP$. Thus P commutes with Q iff $[P, Q] = 0$.
Also, $P \perp Q \;\equiv\; PQ = QP = 0$.

Projectors as quantum propositions

Quantum observables or (projective) measurements are defined by families of projectors $\{P_i\}_i$ with $\sum_i P_i = I$.

These **projective resolutions of the identity** give disjoint cases for the various possible outcomes i.

The commutator $[P, Q] := PQ - QP$. Thus P commutes with Q iff $[P, Q] = 0$.
Also, $P \perp Q \;\equiv\; PQ = QP = 0$.

Given projectors P, Q:

- PQ is a projector iff $[P, Q] = 0$.
- $P + Q$ is a projector iff $P \perp Q$.
- $I - P$ is always a projector.

Projectors as quantum propositions

Quantum observables or (projective) measurements are defined by families of projectors $\{P_i\}_i$ with $\sum_i P_i = I$.

These **projective resolutions of the identity** give disjoint cases for the various possible outcomes i.

The commutator $[P, Q] := PQ - QP$. Thus P commutes with Q iff $[P, Q] = 0$.
Also, $P \perp Q \;\equiv\; PQ = QP = 0$.

Given projectors P, Q:

- PQ is a projector iff $[P, Q] = 0$.
- $P + Q$ is a projector iff $P \perp Q$.
- $I - P$ is always a projector.

Given a projector P, then $\{P, (I - P)\}$ is a projective resolution of the identity. Thus projectors can be viewed as basic **quantum propositions** with operational content.

Projectors and subspaces

Projectors are in bijective correspondence with subspaces:

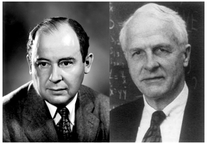
$$\Sigma(P) := \{v \mid P(v) = v\}$$

$$\Sigma(I - P) = \Sigma(P)^{\perp}$$

$$[P, Q] = 0 \;\Rightarrow\; \begin{cases} \Sigma(PQ) & = & \Sigma(P) \cap \Sigma(Q) \\ \Sigma(P + Q - PQ) & = & \Sigma(P) \vee \Sigma(Q) \end{cases}$$

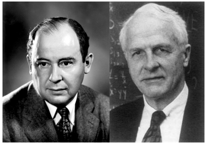
$$P \perp Q \;\Rightarrow\; \Sigma(P + Q) = \Sigma(P) \oplus \Sigma(Q)$$

Background: traditional quantum logic



John von Neumann, in his seminal
Mathematical Foundations of Quantum Mechanics (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^{\dagger}$) and idempotent ($P^2 = P$).

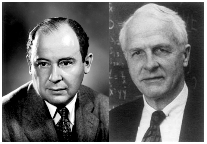
Background: traditional quantum logic



John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Background: traditional quantum logic

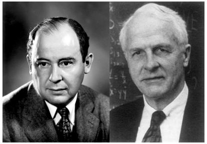


John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Subsequently, Birkhoff and von Neumann, in *The Logic of Quantum Mechanics* (1936), proposed the lattice of closed subspaces as a non-classical logic to serve as the logical foundations of quantum mechanics.

Background: traditional quantum logic



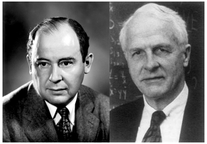
John von Neumann, in his seminal
Mathematical Foundations of Quantum Mechanics (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Subsequently, Birkhoff and von Neumann, in *The Logic of Quantum Mechanics* (1936), proposed the lattice of closed subspaces as a non-classical logic to serve as the logical foundations of quantum mechanics.

- Interpret $\wedge$ (infimum) and $\vee$ (supremum) as logical operations.

Background: traditional quantum logic



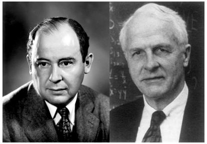
John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Subsequently, Birkhoff and von Neumann, in *The Logic of Quantum Mechanics* (1936), proposed the lattice of closed subspaces as a non-classical logic to serve as the logical foundations of quantum mechanics.

- Interpret $\wedge$ (infimum) and $\vee$ (supremum) as logical operations.
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.

Background: traditional quantum logic



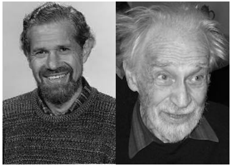
John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Subsequently, Birkhoff and von Neumann, in *The Logic of Quantum Mechanics* (1936), proposed the lattice of closed subspaces as a non-classical logic to serve as the logical foundations of quantum mechanics.

- Interpret $\wedge$ (infimum) and $\vee$ (supremum) as logical operations.
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.
- Only commuting measurements can be performed together.
  So, what is the operational meaning of $p \wedge q$, when p and q **do not commute**?

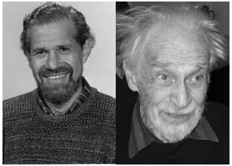
Quantum physics and logic



An alternative approach

Kochen & Specker (1965), '*The problem of hidden variables in quantum mechanics*'.

Quantum physics and logic

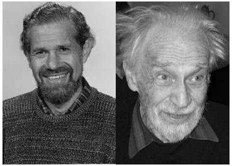


An alternative approach

Kochen & Specker (1965), ‘*The problem of hidden variables in quantum mechanics*’.

- The seminal work on contextuality used **partial Boolean algebras**.
- Only admit physically meaningful operations.
- Represent incompatibility by **partiality**.

Quantum physics and logic



An alternative approach

Kochen & Specker (1965), '*The problem of hidden variables in quantum mechanics*'.

- The seminal work on contextuality used **partial Boolean algebras**.
- Only admit physically meaningful operations.
- Represent incompatibility by **partiality**.

Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

such that every set S of pairwise-commeasurable elements is contained in a set T of pairwise-commeasurable elements which is a Boolean algebra under the restriction of the operations.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

such that every set S of pairwise-commeasurable elements is contained in a set T of pairwise-commeasurable elements which is a Boolean algebra under the restriction of the operations.

The key example: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

such that every set S of pairwise-commeasurable elements is contained in a set T of pairwise-commeasurable elements which is a Boolean algebra under the restriction of the operations.

The key example: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.
Conjunction, i.e. meet of projectors, becomes partial, defined only on **commuting** projectors.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

such that every set S of pairwise-commeasurable elements is contained in a set T of pairwise-commeasurable elements which is a Boolean algebra under the restriction of the operations.

The key example: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.
Conjunction, i.e. meet of projectors, becomes partial, defined only on **commuting** projectors.

Morphisms of pBAs are maps preserving commeasurability, and the operations wherever defined. This gives the category **pBA**.

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathsf{P}(\mathcal{H})$ its pBA of projectors.

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathsf{P}(\mathcal{H})$ its pBA of projectors.

$$\boxed{\text{There is } \textbf{no} \text{ pBA homomorphism } \mathbf{P}(\mathcal{H}) \longrightarrow \mathbf{2}.}$$

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathsf{P}(\mathcal{H})$ its pBA of projectors.

$$\boxed{\text{There is } \textbf{no} \text{ pBA homomorphism } \mathbf{P}(\mathcal{H}) \longrightarrow \mathbf{2}.}$$

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathsf{P}(\mathcal{H})$ its pBA of projectors.

$$\boxed{\text{There is \textbf{no} pBA homomorphism } \mathbf{P}(\mathcal{H}) \longrightarrow \mathbf{2}.}$$

- No assignment of truth values to all propositions that respects the logical operations on jointly testable propositions.

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathrm{P}(\mathcal{H})$ its pBA of projectors.

$$\boxed{\text{There is } \textbf{no} \text{ pBA homomorphism } \mathbf{P}(\mathcal{H}) \longrightarrow \mathbf{2}.}$$

- No assignment of truth values to all propositions that respects the logical operations on jointly testable propositions.
- Spectrum of a pBA cannot have *points*. . .

Conditions of impossible experience

Conditions of impossible experience

Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to **2**)*
2. *For some **propositional contradiction** $\varphi(\vec{x})$ and assignment $\vec{x} \mapsto \vec{a}$,*

$$A \models \varphi(\vec{a})$$

Conditions of impossible experience

Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to* **2***)*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$*,*

$$A \models \varphi(\vec{a})$$

Thus the event algebra $P(\mathcal{H})$ of quantum mechanics cannot be interpreted globally in a consistent fashion.

Conditions of impossible experience

Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to* **2***)*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$*,*

$$A \models \varphi(\vec{a})$$

Thus the event algebra $\mathrm{P}(\mathcal{H})$ of quantum mechanics cannot be interpreted globally in a consistent fashion.

Our local observations – **real observations** of **real measurements** – cannot be pieced together globally by reference to a single underlying objective reality. The values that they reveal are inherently contextual.

Conditions of impossible experience

Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to **2**)*
2. *For some **propositional contradiction** $\varphi(\vec{x})$ and assignment $\vec{x} \mapsto \vec{a}$,*

$$A \models \varphi(\vec{a})$$

Thus the event algebra $\mathrm{P}(\mathcal{H})$ of quantum mechanics cannot be interpreted globally in a consistent fashion.

Our local observations – **real observations** of **real measurements** – cannot be pieced together globally by reference to a single underlying objective reality. The values that they reveal are inherently contextual.

How can the world be this way? Still an ongoing debate, an enduring mystery . . .

Contrast with Intuitionistic logic

Say that a **classical contradiction** is a propositional formula φ such that $\mathsf{CL} \vdash \neg\varphi$.

Theorem

If $\mathsf{CL} \vdash \neg\varphi$*, then* $\mathsf{IL} \vdash \neg\varphi$.

Proof.

If $\mathsf{CL} \vdash \neg\varphi$, then by Glivenko's theorem, $\mathsf{IL} \vdash \neg\neg\neg\varphi$. Since $\mathsf{IL} \vdash \neg\neg\neg p \longrightarrow \neg p$, it follows that $\mathsf{IL} \vdash \neg\varphi$. $\square$

Thus **every classical contradiction is an intuitionistic contradiction**.

Contrast with Intuitionistic logic

Say that a **classical contradiction** is a propositional formula φ such that $\mathsf{CL} \vdash \neg\varphi$.

Theorem

If $\mathsf{CL} \vdash \neg\varphi$*, then* $\mathsf{IL} \vdash \neg\varphi$*.*

Proof.

If $\mathsf{CL} \vdash \neg\varphi$, then by Glivenko's theorem, $\mathsf{IL} \vdash \neg\neg\neg\varphi$. Since $\mathsf{IL} \vdash \neg\neg\neg p \longrightarrow \neg p$, it follows that $\mathsf{IL} \vdash \neg\varphi$. $\square$

Thus **every classical contradiction is an intuitionistic contradiction**.

As a corollary, we obtain:

Theorem

A classical contradiction cannot be satisfied in any sound semantics for intuitionistic logic.

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

It is a standard fact that every finitely-generated boolean algebra is finite.

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

It is a standard fact that every finitely-generated boolean algebra is finite.

Conway and Kochen (2002) show the following:

Theorem

In $\mathrm{P}(\mathbb{C}^4)$, there is a set of five projectors (local Paulis) which generate a ***uniformly dense (infinite) subalgebra****.*

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

It is a standard fact that every finitely-generated boolean algebra is finite.

Conway and Kochen (2002) show the following:

Theorem

In $\mathrm{P}(\mathbb{C}^4)$*, there is a set of five projectors (local Paulis) which generate a* ***uniformly dense (infinite) subalgebra****.*

Some elaborate geometry and algebra is used to show this.

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

It is a standard fact that every finitely-generated boolean algebra is finite.

Conway and Kochen (2002) show the following:

Theorem

In $\mathsf{P}(\mathbb{C}^4)$*, there is a set of five projectors (local Paulis) which generate a* ***uniformly dense (infinite) subalgebra****.*

Some elaborate geometry and algebra is used to show this.

Is there a “logical” proof?

The category **pBA**

In Heunen and van der Berg, **Non-commutativity as a colimit** (2012), it is shown that every partial Boolean algebra is the colimit of its Boolean subalgebras.

The category **pBA**

In Heunen and van der Berg, **Non-commutativity as a colimit** (2012), it is shown that every partial Boolean algebra is the colimit of its Boolean subalgebras.

Coproducts have a simple direct description. The coproduct $A \oplus B$ of partial Boolean algebras A, B is their disjoint union with 0_A identified with 0_B, and 1_A identified with 1_B. Other than these identifications, no commeasurability holds between elements of A and elements of B.

The category **pBA**

In Heunen and van der Berg, **Non-commutativity as a colimit** (2012), it is shown that every partial Boolean algebra is the colimit of its Boolean subalgebras.

Coproducts have a simple direct description. The coproduct $A \oplus B$ of partial Boolean algebras A, B is their disjoint union with 0_A identified with 0_B, and 1_A identified with 1_B. Other than these identifications, no commeasurability holds between elements of A and elements of B.

N.B. This is very different to coproducts in **BA**!

The category **pBA**

In Heunen and van der Berg, **Non-commutativity as a colimit** (2012), it is shown that every partial Boolean algebra is the colimit of its Boolean subalgebras.

Coproducts have a simple direct description. The coproduct $A \oplus B$ of partial Boolean algebras A, B is their disjoint union with 0_A identified with 0_B, and 1_A identified with 1_B. Other than these identifications, no commeasurability holds between elements of A and elements of B.

N.B. This is very different to coproducts in **BA**!

By contrast, coequalisers, and general colimits, are shown to exist by Heunen and van der Berg by an appeal to the Adjoint Functor Theorem. One of our contributions is to give an explicit construction of the needed colimits,.

The category **pBA**

In Heunen and van der Berg, **Non-commutativity as a colimit** (2012), it is shown that every partial Boolean algebra is the colimit of its Boolean subalgebras.

Coproducts have a simple direct description. The coproduct $A \oplus B$ of partial Boolean algebras A, B is their disjoint union with 0_A identified with 0_B, and 1_A identified with 1_B. Other than these identifications, no commeasurability holds between elements of A and elements of B.

N.B. This is very different to coproducts in **BA**!

By contrast, coequalisers, and general colimits, are shown to exist by Heunen and van der Berg by an appeal to the Adjoint Functor Theorem. One of our contributions is to give an explicit construction of the needed colimits,.

More generally, we use this approach to prove the following result, which freely generates from a given partial Boolean algebra a new one where prescribed additional commeasurability relations are enforced between its elements.

Theorem

Given a partial Boolean algebra A and a binary relation $\circledcirc$ on A, there is a partial Boolean algebra $A[\circledcirc]$ such that:

- *There is a* **pBA***-morphism $\eta : A \longrightarrow A[\circledcirc]$ such that $a \circledcirc b \;\Rightarrow\; \eta(a) \odot_{A[\circledcirc]} \eta(b)$.*
- *For every partial Boolean algebra B and* **pBA***-morphism $h : A \longrightarrow B$ such that $a \circledcirc b \;\Rightarrow\; h(a) \odot_B h(b)$, there is a unique homomorphism $\hat{h} : A[\circledcirc] \longrightarrow B$ such that*

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "h"'] & A[\circledcirc] \arrow[d, "\hat{h}"] \\ & B \end{tikzcd}$$

This result is proved constructively, by giving proof rules for commeasurability and equivalence relations over a set of syntactic terms generated from A. (In fact, we start with a set of "pre-terms", and also give rules for definedness).

The inductive construction

$$\frac{a \in A}{\imath(a)\downarrow} \qquad \frac{a \odot_A b}{\imath(a) \odot \imath(b)} \qquad \frac{a \circledcirc b}{\imath(a) \odot \imath(b)}$$

$$\frac{}{0 \equiv \imath(0_A),\ 1 \equiv \imath(1_A),\ \neg\imath(a) \equiv \imath(\neg_A a)}$$

$$\frac{a \odot_A b}{\imath(a) \wedge \imath(b) \equiv \imath(a \wedge_A b),\ \imath(a) \vee \imath(b) \equiv \imath(a \vee_A b)}$$

$$\frac{}{0\downarrow,\ 1\downarrow} \qquad \frac{t \odot u}{t \wedge u\downarrow,\ t \vee u\downarrow} \qquad \frac{t\downarrow}{\neg t\downarrow}$$

$$\frac{t\downarrow}{t \odot t,\ t \odot 0,\ t \odot 1} \qquad \frac{t \odot u}{u \odot t} \qquad \frac{t \odot u,\ t \odot v,\ u \odot v}{t \wedge u \odot v,\ t \vee u \odot v} \qquad \frac{t \odot u}{\neg t \odot u}$$

$$\frac{t\downarrow}{t \equiv t} \qquad \frac{t \equiv u}{u \equiv v} \qquad \frac{t \equiv u,\ u \equiv v}{t \equiv v} \qquad \frac{t \equiv u,\ u \odot v}{t \odot v}$$

$$\frac{\varphi(\vec{x}) \equiv_{\mathrm{Bool}} \psi(\vec{x}),\ \bigwedge_{i,j} v_i \odot v_j}{\varphi(\vec{v}) \equiv \psi(\vec{v})} \qquad \frac{t \equiv t',\ u \equiv u',\ t \odot u}{t \wedge u \equiv t' \wedge u',\ t \vee u \equiv t' \vee u'} \qquad \frac{t \equiv u}{\neg t \equiv \neg u}$$

Coequalisers and colimits

A variation of this construction is also useful, where instead of just forcing commeasurability, one forces equality by the additional rule

$$\frac{a \circledcirc a'}{\imath(a) \equiv \imath(a')}$$

This builds a pBA $A[\circledcirc, \equiv]$.

Theorem

Let $h : A \longrightarrow B$ be a **pBA**-*morphism such that $a \circledcirc a' \;\Rightarrow\; h(a) = h(a')$. Then there is a unique* **pBA**-*morphism $\hat{h} : A[\circledcirc, \equiv] \longrightarrow B$ such that $h = \hat{h} \circ \eta$.*

This result can be used to give an explicit construction of coequalisers, and hence general colimits, in **pBA**.

An apparent contradiction

BA is a full subcategory of **pBA**. We know from (Heunen and van den Berg) that A is the colimit in **pBA** of its boolean subalgebras. Now let B be the colimit in **BA** of the same diagram D of boolean subalgebras of A and the inclusions between them.

An apparent contradiction

BA is a full subcategory of **pBA**. We know from (Heunen and van den Berg) that A is the colimit in **pBA** of its boolean subalgebras. Now let B be the colimit in **BA** of the same diagram D of boolean subalgebras of A and the inclusions between them.

Then the cone from D to B is also a cone in **pBA**, hence there is a mediating morphism from A to B!

An apparent contradiction

BA is a full subcategory of **pBA**. We know from (Heunen and van den Berg) that A is the colimit in **pBA** of its boolean subalgebras. Now let B be the colimit in **BA** of the same diagram D of boolean subalgebras of A and the inclusions between them.

Then the cone from D to B is also a cone in **pBA**, hence there is a mediating morphism from A to B!

To resolve the apparent contradiction, note that **BA** is an equational variety of algebras over **Set**.

An apparent contradiction

BA is a full subcategory of **pBA**. We know from (Heunen and van den Berg) that A is the colimit in **pBA** of its boolean subalgebras. Now let B be the colimit in **BA** of the same diagram D of boolean subalgebras of A and the inclusions between them.

Then the cone from D to B is also a cone in **pBA**, hence there is a mediating morphism from A to B!

To resolve the apparent contradiction, note that **BA** is an equational variety of algebras over **Set**.

As such, it is complete and cocomplete, but it also admits the one-element algebra **1**, in which $0 = 1$. Note that **1** does **not** have a homomorphism to **2**.

An apparent contradiction

BA is a full subcategory of **pBA**. We know from (Heunen and van den Berg) that A is the colimit in **pBA** of its boolean subalgebras. Now let B be the colimit in **BA** of the same diagram D of boolean subalgebras of A and the inclusions between them.

Then the cone from D to B is also a cone in **pBA**, hence there is a mediating morphism from A to B!

To resolve the apparent contradiction, note that **BA** is an equational variety of algebras over **Set**.

As such, it is complete and cocomplete, but it also admits the one-element algebra **1**, in which $0 = 1$. Note that **1** does **not** have a homomorphism to **2**.

In the case of a partial Boolean algebra with the K-S property of not having a homomorphism to **2**, the colimit of its diagram of boolean subalgebras must be **1**.

KS-property and colimits

We can turn this into a theorem:

Theorem

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property.*
2. *The colimit of the diagram of boolean subalgebras of A in* $\mathbf{BA}$ *is* $\mathbf{1}$.

KS-property and colimits

We can turn this into a theorem:

Theorem

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property.*
2. *The colimit of the diagram of boolean subalgebras of A in* **BA** *is* **1**.

In fact, we can formulate the K-S property directly for diagrams of Boolean algebras, without referring to partial boolean algebras at all.

KS-property and colimits

We can turn this into a theorem:

Theorem

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property.*
2. *The colimit of the diagram of boolean subalgebras of A in $\mathbf{BA}$ is $\mathbf{1}$.*

In fact, we can formulate the K-S property directly for diagrams of Boolean algebras, without referring to partial boolean algebras at all.

We say that a diagram in $\mathbf{BA}$ is K-S if its colimit in $\mathbf{BA}$ is $\mathbf{1}$.

KS-property and colimits

We can turn this into a theorem:

Theorem

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property.*
2. *The colimit of the diagram of boolean subalgebras of A in* **BA** *is* **1**.

In fact, we can formulate the K-S property directly for diagrams of Boolean algebras, without referring to partial boolean algebras at all.

We say that a diagram in **BA** is K-S if its colimit in **BA** is **1**.

We could say that such a diagram is “implicitly contradictory”, and in trying to combine all the information in a colimit, we obtain the manifestly contradictory **1**.

KS-property and colimits

We can turn this into a theorem:

Theorem

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property.*
2. *The colimit of the diagram of boolean subalgebras of A in* **BA** *is* **1**.

In fact, we can formulate the K-S property directly for diagrams of Boolean algebras, without referring to partial boolean algebras at all.

We say that a diagram in **BA** is K-S if its colimit in **BA** is **1**.

We could say that such a diagram is "implicitly contradictory", and in trying to combine all the information in a colimit, we obtain the manifestly contradictory **1**.

A partial Boolean algebra with the K-S property – such as $\mathrm{P}(\mathcal{H})$ – holds this implicitly contradictory information together in a single structure.

KS-property and free extensions

KS-property and free extensions

We now consider the relationship of the K-S property to the free extension of partial Boolean algebras by a relation, as just described.

KS-property and free extensions

We now consider the relationship of the K-S property to the free extension of partial Boolean algebras by a relation, as just described.

Proposition

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property.*
2. $A[A^2] = \mathbf{1}$.

KS-property and free extensions

We now consider the relationship of the K-S property to the free extension of partial Boolean algebras by a relation, as just described.

Proposition

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property.*
2. $A[A^2] = \mathbf{1}$.

Proof.

Firstly, all elements are commeasurable in $A[A^2]$, so it is a Boolean algebra. Moreover, there is a morphism $\eta : A \longrightarrow A[A^2]$. Thus if A is K-S, we must have $A[A^2] = \mathbf{1}$.

KS-property and free extensions

We now consider the relationship of the K-S property to the free extension of partial Boolean algebras by a relation, as just described.

Proposition

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property.*
2. $A[A^2] = \mathbf{1}$.

Proof.

Firstly, all elements are commeasurable in $A[A^2]$, so it is a Boolean algebra. Moreover, there is a morphism $\eta : A \longrightarrow A[A^2]$. Thus if A is K-S, we must have $A[A^2] = \mathbf{1}$.

Conversely, suppose that $A[A^2] = \mathbf{1}$, and there is a morphism $A \longrightarrow B$ to a Boolean algebra A. By the universal property of $A[A^2]$, there is a morphism $A[A^2] \longrightarrow B$, and since $A[A^2] = \mathbf{1}$, we must have $B = \mathbf{1}$. Thus A is K-S. $\square$

Tensor product and the emergence of non-classicality

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathsf{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathrm{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Note that $\mathbf{P}(\mathbb{C}^2) \cong \bigoplus_{i \in I} \mathbf{4}_i$, where I is a set of the power of the continuum, and each $\mathbf{4}_i$ is the four-element Boolean algebra.

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathsf{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Note that $\mathbf{P}(\mathbb{C}^2) \cong \bigoplus_{i \in I} \mathbf{4}_i$, where I is a set of the power of the continuum, and each $\mathbf{4}_i$ is the four-element Boolean algebra.

One of the key points at which non-classicality emerges in quantum theory is the passage from $\mathsf{P}(\mathbb{C}^2)$, which **does not** have the K–S property, to $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, which **does**.

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathsf{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Note that $\mathbf{P}(\mathbb{C}^2) \cong \bigoplus_{i \in I} \mathbf{4}_i$, where I is a set of the power of the continuum, and each $\mathbf{4}_i$ is the four-element Boolean algebra.

One of the key points at which non-classicality emerges in quantum theory is the passage from $\mathsf{P}(\mathbb{C}^2)$, which **does not** have the K–S property, to $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, which **does**.

Can we capture the Hilbert space tensor product in logical form?

Question

Is there a monoidal structure $\circledast$ on the category **pBA** *such that the functor* $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$ *is* ***strong monoidal*** *with respect to this structure, i.e. such that* $\mathsf{P}(\mathcal{H}) \circledast \mathsf{P}(\mathcal{K}) \cong \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$*?*

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathsf{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Note that $\mathbf{P}(\mathbb{C}^2) \cong \bigoplus_{i \in I} \mathbf{4}_i$, where I is a set of the power of the continuum, and each $\mathbf{4}_i$ is the four-element Boolean algebra.

One of the key points at which non-classicality emerges in quantum theory is the passage from $\mathsf{P}(\mathbb{C}^2)$, which **does not** have the K–S property, to $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, which **does**.

Can we capture the Hilbert space tensor product in logical form?

Question

Is there a monoidal structure $\circledast$ on the category **pBA** *such that the functor* $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$ *is* ***strong monoidal*** *with respect to this structure, i.e. such that* $\mathsf{P}(\mathcal{H}) \circledast \mathsf{P}(\mathcal{K}) \cong \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$*?*

A positive answer to this question would offer a complete logical characterisation of the Hilbert space tensor product, and provide an important step towards giving logical foundations for quantum theory in a form useful for quantum information and computation.

Tensor products of partial Boolean algebras

Tensor products of partial Boolean algebras

In (Heunen and van den Berg), it is shown that **pBA** has a monoidal structure, with $A \otimes B$ given by the colimit of the family of $C + D$, as C ranges over Boolean subalgebras of A, D ranges over Boolean subalgebras of B, and $C + D$ is the coproduct of Boolean algebras.

Tensor products of partial Boolean algebras

In (Heunen and van den Berg), it is shown that **pBA** has a monoidal structure, with $A \otimes B$ given by the colimit of the family of $C + D$, as C ranges over Boolean subalgebras of A, D ranges over Boolean subalgebras of B, and $C + D$ is the coproduct of Boolean algebras.

The tensor product there is not constructed explicitly: it relies on the existence of colimits in **pBA**, which is proved by an appeal to the Adjoint Functor Theorem.

Tensor products of partial Boolean algebras

In (Heunen and van den Berg), it is shown that **pBA** has a monoidal structure, with $A \otimes B$ given by the colimit of the family of $C + D$, as C ranges over Boolean subalgebras of A, D ranges over Boolean subalgebras of B, and $C + D$ is the coproduct of Boolean algebras.

The tensor product there is not constructed explicitly: it relies on the existence of colimits in **pBA**, which is proved by an appeal to the Adjoint Functor Theorem.

Our Theorem 5 allows us to give an explicit description of this construction using generators and relations.

Proposition

Let A and B be partial Boolean algebras. Then

$$A \otimes B \;\cong\; (A \oplus B)[\odot]$$

where $\odot$ is the relation on the carrier set of $A \oplus B$ given by $\imath(a) \odot \jmath(b)$ for all $a \in A$ and $b \in B$.

Limitations of this tensor product

Limitations of this tensor product

There is a lax monoidal functor $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$, which takes a Hilbert space to its projectors, viewed as a partial Boolean algebra, with an embedding $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K}) \longrightarrow \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the evident embeddings of $\mathsf{P}(\mathcal{H})$ and $\mathsf{P}(\mathcal{K})$ into $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$), given by $p \longmapsto p \otimes 1$, $q \longmapsto 1 \otimes q$.

Limitations of this tensor product

There is a lax monoidal functor $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$, which takes a Hilbert space to its projectors, viewed as a partial Boolean algebra, with an embedding $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K}) \longrightarrow \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the evident embeddings of $\mathsf{P}(\mathcal{H})$ and $\mathsf{P}(\mathcal{K})$ into $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$), given by $p \longmapsto p \otimes 1$, $q \longmapsto 1 \otimes q$.

It is easy to see that this embedding is far from being surjective. For example, if we take $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$, then there are (many) two-valued homomorphisms on $A = \mathsf{P}(\mathbb{C}^2)$, which lift to two-valued homomorphisms on $A \otimes A$. However, by the Kochen–Specker theorem, there is no such homomorphism on $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

Limitations of this tensor product

There is a lax monoidal functor $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$, which takes a Hilbert space to its projectors, viewed as a partial Boolean algebra, with an embedding $\mathrm{P}(\mathcal{H}) \otimes \mathrm{P}(\mathcal{K}) \longrightarrow \mathrm{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the evident embeddings of $\mathrm{P}(\mathcal{H})$ and $\mathrm{P}(\mathcal{K})$ into $\mathrm{P}(\mathcal{H} \otimes \mathcal{K})$), given by $p \longmapsto p \otimes 1$, $q \longmapsto 1 \otimes q$.

It is easy to see that this embedding is far from being surjective. For example, if we take $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$, then there are (many) two-valued homomorphisms on $A = \mathrm{P}(\mathbb{C}^2)$, which lift to two-valued homomorphisms on $A \otimes A$. However, by the Kochen−Specker theorem, there is no such homomorphism on $\mathrm{P}(\mathbb{C}^4) = \mathrm{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

Interestingly, in (Kochen 2015) it is shown that the images of $\mathrm{P}(\mathcal{H})$ and $\mathrm{P}(\mathcal{K})$, for any finite-dimensional $\mathcal{H}$ and $\mathcal{K}$, generate $\mathrm{P}(\mathcal{H} \otimes \mathcal{K})$. This is used there to justify the claim contradicted by the previous paragraph. The gap in the argument is that more relations hold in $\mathrm{P}(\mathcal{H} \otimes \mathcal{K})$ than in $\mathrm{P}(\mathcal{H}) \otimes \mathrm{P}(\mathcal{K})$.

Limitations of this tensor product

There is a lax monoidal functor $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$, which takes a Hilbert space to its projectors, viewed as a partial Boolean algebra, with an embedding $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K}) \longrightarrow \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the evident embeddings of $\mathsf{P}(\mathcal{H})$ and $\mathsf{P}(\mathcal{K})$ into $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$), given by $p \longmapsto p \otimes 1$, $q \longmapsto 1 \otimes q$.

It is easy to see that this embedding is far from being surjective. For example, if we take $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$, then there are (many) two-valued homomorphisms on $A = \mathsf{P}(\mathbb{C}^2)$, which lift to two-valued homomorphisms on $A \otimes A$. However, by the Kochen–Specker theorem, there is no such homomorphism on $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

Interestingly, in (Kochen 2015) it is shown that the images of $\mathsf{P}(\mathcal{H})$ and $\mathsf{P}(\mathcal{K})$, for any finite-dimensional $\mathcal{H}$ and $\mathcal{K}$, generate $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$. This is used there to justify the claim contradicted by the previous paragraph. The gap in the argument is that more relations hold in $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ than in $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K})$.

Nevertheless, this result is very suggestive. It poses the challenge of finding a stronger notion of tensor product.

Towards a more expressive tensor product

Towards a more expressive tensor product

An important property satisfied by the rules in Table 1 as applied in constructing $A \otimes B$ is that, if $t{\downarrow}$ can be derived, then $u{\downarrow}$ can be derived for every subterm u of t. This appears to be too strong a constraint to capture the full logic of the Hilbert space tensor product.

Towards a more expressive tensor product

An important property satisfied by the rules in Table 1 as applied in constructing $A \otimes B$ is that, if $t{\downarrow}$ can be derived, then $u{\downarrow}$ can be derived for every subterm u of t. This appears to be too strong a constraint to capture the full logic of the Hilbert space tensor product.

To see why this is an issue, consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$. To ensure in general that they commute, we need the conjunctive requirement that p_1 commutes with q_1, **and** p_2 commutes with q_2.

Towards a more expressive tensor product

An important property satisfied by the rules in Table 1 as applied in constructing $A \otimes B$ is that, if $t\downarrow$ can be derived, then $u\downarrow$ can be derived for every subterm u of t. This appears to be too strong a constraint to capture the full logic of the Hilbert space tensor product.

To see why this is an issue, consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$. To ensure in general that they commute, we need the conjunctive requirement that p_1 commutes with q_1, **and** p_2 commutes with q_2.

However, to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$. If we establish orthogonality in this way, we are entitled to conclude that $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are commeasurable, even though (say) p_2 and q_2 are not.

Towards a more expressive tensor product

An important property satisfied by the rules in Table 1 as applied in constructing $A \otimes B$ is that, if $t\downarrow$ can be derived, then $u\downarrow$ can be derived for every subterm u of t. This appears to be too strong a constraint to capture the full logic of the Hilbert space tensor product.

To see why this is an issue, consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$. To ensure in general that they commute, we need the conjunctive requirement that p_1 commutes with q_1, **and** p_2 commutes with q_2.

However, to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$. If we establish orthogonality in this way, we are entitled to conclude that $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are commeasurable, even though (say) p_2 and q_2 are not.

Indeed, the idea that propositions can be defined on quantum systems even though subexpressions are not is emphasized by Kochen.

Logical exclusivity principle

The basic ingredient is a notion of exclusivity between events (or elements) of a partial Boolean algebra.

Logical exclusivity principle

The basic ingredient is a notion of exclusivity between events (or elements) of a partial Boolean algebra.

Definition

Let A be a partial Boolean algebra. Two elements $a, b \in A$ are said to be **exclusive**, written $a \perp b$, if there is a $c \in A$ such that $a \leq c$ and $b \leq \neg c$.

Logical exclusivity principle

The basic ingredient is a notion of exclusivity between events (or elements) of a partial Boolean algebra.

Definition

Let A be a partial Boolean algebra. Two elements $a, b \in A$ are said to be **exclusive**, written $a \perp b$, if there is a $c \in A$ such that $a \leq c$ and $b \leq \neg c$.

Note that $x \leq y$ in a pBA means that $x \odot y$ and $x \wedge y = x$.

Logical exclusivity principle

The basic ingredient is a notion of exclusivity between events (or elements) of a partial Boolean algebra.

Definition

Let A be a partial Boolean algebra. Two elements $a, b \in A$ are said to be **exclusive**, written $a \perp b$, if there is a $c \in A$ such that $a \leq c$ and $b \leq \neg c$.

Note that $x \leq y$ in a pBA means that $x \odot y$ and $x \wedge y = x$.

Thus $a \perp b$ is a weaker requirement than $a \wedge b = 0$, although the two would be equivalent in a Boolean algebra. The point is that, in a general partial Boolean algebra, one might have exclusive events that are not commeasurable (and for which, therefore, the $\wedge$ operation is not defined).

Logical exclusivity principle

The basic ingredient is a notion of exclusivity between events (or elements) of a partial Boolean algebra.

Definition

Let A be a partial Boolean algebra. Two elements $a, b \in A$ are said to be **exclusive**, written $a \perp b$, if there is a $c \in A$ such that $a \leq c$ and $b \leq \neg c$.

Note that $x \leq y$ in a pBA means that $x \odot y$ and $x \wedge y = x$.

Thus $a \perp b$ is a weaker requirement than $a \wedge b = 0$, although the two would be equivalent in a Boolean algebra. The point is that, in a general partial Boolean algebra, one might have exclusive events that are not commeasurable (and for which, therefore, the $\wedge$ operation is not defined).

Definition

A partial Boolean algebra A is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are logically exclusive are also commeasurable, i.e. if $\perp \subseteq \odot$.
We write **epBA** for the full subcategory of **pBA** whose objects are partial Boolean algebras satisfying LEP.

Logical exclusivity and transitivity

The logical exclusivity principle turns out to be equivalent to the following notion of transitivity.

Definition

A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.

Logical exclusivity and transitivity

The logical exclusivity principle turns out to be equivalent to the following notion of transitivity.

Definition

A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.

Transitivity can fail in general for a partial Boolean algebra, since one need not have $a \odot c$ under the stated hypotheses. Note that the relation $\leq$ on a partial Boolean algebra is always reflexive and anti-symmetric, so this condition is equivalent to $\leq$ being a partial order (globally) on A.

Logical exclusivity and transitivity

The logical exclusivity principle turns out to be equivalent to the following notion of transitivity.

Definition

A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.

Transitivity can fail in general for a partial Boolean algebra, since one need not have $a \odot c$ under the stated hypotheses. Note that the relation $\leq$ on a partial Boolean algebra is always reflexive and anti-symmetric, so this condition is equivalent to $\leq$ being a partial order (globally) on A.

A partial Boolean algebra of the form $\mathrm{P}(\mathcal{H})$ is always transitive.

Logical exclusivity and transitivity

The logical exclusivity principle turns out to be equivalent to the following notion of transitivity.

Definition

A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.

Transitivity can fail in general for a partial Boolean algebra, since one need not have $a \odot c$ under the stated hypotheses. Note that the relation $\leq$ on a partial Boolean algebra is always reflexive and anti-symmetric, so this condition is equivalent to $\leq$ being a partial order (globally) on A.

A partial Boolean algebra of the form $\mathrm{P}(\mathcal{H})$ is always transitive.

Proposition

Let A be a partial Boolean algebra. Then it satisfies LEP if and only if it is transitive.

Logical exclusivity and transitivity

The logical exclusivity principle turns out to be equivalent to the following notion of transitivity.

Definition

A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.

Transitivity can fail in general for a partial Boolean algebra, since one need not have $a \odot c$ under the stated hypotheses. Note that the relation $\leq$ on a partial Boolean algebra is always reflexive and anti-symmetric, so this condition is equivalent to $\leq$ being a partial order (globally) on A.

A partial Boolean algebra of the form $\mathrm{P}(\mathcal{H})$ is always transitive.

Proposition

Let A be a partial Boolean algebra. Then it satisfies LEP if and only if it is transitive.

As an immediate consequence, any $\mathrm{P}(\mathcal{H})$ satisfies LEP.

A reflective adjunction for logical exclusivity

We can of course form the partial Boolean algebra $A[\perp]$. While the exclusivity principle holds for all its elements in the image of $\eta : A \longrightarrow A[\perp]$, it may fail to hold for other elements in $A[\perp]$.

A reflective adjunction for logical exclusivity

We can of course form the partial Boolean algebra $A[\perp]$. While the exclusivity principle holds for all its elements in the image of $\eta : A \longrightarrow A[\perp]$, it may fail to hold for other elements in $A[\perp]$.

However, we can adapt our construction to show that one can freely generate, from any given partial Boolean algebra, a new partial Boolean algebra satisfying LEP.

A reflective adjunction for logical exclusivity

We can of course form the partial Boolean algebra $A[\perp]$. While the exclusivity principle holds for all its elements in the image of $\eta : A \longrightarrow A[\perp]$, it may fail to hold for other elements in $A[\perp]$.

However, we can adapt our construction to show that one can freely generate, from any given partial Boolean algebra, a new partial Boolean algebra satisfying LEP.

This LEP-isation is analogous to e.g. the way one can 'abelianise' any group, or use Stone–Čech compactification to form a compact Hausdorff space from any topological space.

Theorem

The category **epBA** *is a reflective subcategory of* **pBA**, *i.e. the inclusion functor* $I : \mathbf{epBA} \longrightarrow \mathbf{pBA}$ *has a left adjoint* $X : \mathbf{pBA} \longrightarrow \mathbf{epBA}$. *Concretely, to any partial Boolean algebra A, we can associate a Boolean algebra* $X(A) = A[\perp]^*$ *which satisfies LEP such that:*

- *there is a homomorphism* $\eta : A \longrightarrow A[\perp]^*$;
- *for any homomorphism* $h : A \longrightarrow B$ *where B is a partial Boolean algebra B satisfying LEP, there is a unique homomorphism* $\hat{h} : A[\perp]^* \longrightarrow B$ *such that:*

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "h"'] & A[\perp]^* \arrow[d, "\hat{h}"] \\ & B \end{tikzcd}$$

Theorem

The category **epBA** *is a reflective subcategory of* **pBA**, *i.e. the inclusion functor I :* **epBA** $\longrightarrow$ **pBA** *has a left adjoint X :* **pBA** $\longrightarrow$ **epBA**. *Concretely, to any partial Boolean algebra A, we can associate a Boolean algebra $X(A) = A[\perp]^*$ which satisfies LEP such that:*

- *there is a homomorphism $\eta : A \longrightarrow A[\perp]^*$;*
- *for any homomorphism $h : A \longrightarrow B$ where B is a partial Boolean algebra B satisfying LEP, there is a unique homomorphism $\hat{h} : A[\perp]^* \longrightarrow B$ such that:*

$$
\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[dr, "h"'] & A[\perp]^* \arrow[d, "\hat{h}"] \\
 & B
\end{tikzcd}
$$

The proof of this result follows from a simple adaptation of the proof of Theorem 5, namely adding the following rule to the inductive system presented in Table 1:

$$\frac{u \wedge t \equiv u, \ v \wedge \neg t \equiv v}{u \odot v}$$

Logical exclusivity tensor product

Logical exclusivity tensor product

We can define a stronger tensor product by forcing logical exclusivity to hold.

Logical exclusivity tensor product

We can define a stronger tensor product by forcing logical exclusivity to hold.

This amounts to composing with the reflection to **epBA**; $\boxtimes := X \circ \otimes$. Explicitly, we define the logical exclusivity tensor product by

$$A \boxtimes B = (A \otimes B)[\perp]^* = (A \oplus B)[\oplus][\perp]^*.$$

Logical exclusivity tensor product

We can define a stronger tensor product by forcing logical exclusivity to hold.

This amounts to composing with the reflection to **epBA**; $\boxtimes := X \circ \otimes$. Explicitly, we define the logical exclusivity tensor product by

$$A \boxtimes B = (A \otimes B)[\perp]^* = (A \oplus B)[\varobar][\perp]^*.$$

This is sound for the Hilbert space model. More precisely, P is still a lax monoidal functor with respect to this tensor product.

Logical exclusivity tensor product

We can define a stronger tensor product by forcing logical exclusivity to hold.

This amounts to composing with the reflection to **epBA**; $\boxtimes := X \circ \otimes$. Explicitly, we define the logical exclusivity tensor product by

$$A \boxtimes B = (A \otimes B)[\perp]^* = (A \oplus B)[\odot][\perp]^*.$$

This is sound for the Hilbert space model. More precisely, P is still a lax monoidal functor with respect to this tensor product.

How close does it it get us to the full Hilbert space tensor product?

KS-faithfulness of extensions

KS-faithfulness of extensions

We can ask generally if extending commeasurability by some relation R can induce the K-S property in $A[R]$ when it did not hold in A?

KS-faithfulness of extensions

We can ask generally if extending commeasurability by some relation R can induce the K-S property in $A[R]$ when it did not hold in A?
In fact, it is easily seen that this can never happen.

KS-faithfulness of extensions

We can ask generally if extending commeasurability by some relation R can induce the K-S property in $A[R]$ when it did not hold in A?
In fact, it is easily seen that this can never happen.

Theorem (K-S faithfulness of extensions)

Let A be a partial Boolean algebra, and $R \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[R]$ is K-S.

KS-faithfulness of extensions

We can ask generally if extending commeasurability by some relation R can induce the K-S property in $A[R]$ when it did not hold in A?
In fact, it is easily seen that this can never happen.

Theorem (K-S faithfulness of extensions)

Let A be a partial Boolean algebra, and $R \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[R]$ is K-S.

Proof.

If A is not K-S, it has a homomorphism to a non-trivial Boolean algebra B. By the universal property of $A[R]$, there is a homomorphism $\hat{h} : A[R] \longrightarrow B$. Thus $A[R]$ is not K-S. Conversely, if there is a morphism $k : A[R] \longrightarrow B$ to a non-trivial Boolean algebra B, then $k \circ \eta : A \longrightarrow B$, so A is not K-S. $\square$

Tensor products

We can apply this in particular to the tensor product.

Tensor products

We can apply this in particular to the tensor product.

Corollary

If A and B are not K-S, then neither is $A \otimes B[\perp]^k$.

Tensor products

We can apply this in particular to the tensor product.

Corollary

If A and B are not K-S, then neither is $A \otimes B[\perp]^k$.

Proof.

If A and B are not K-S, they have homomorphisms to $\mathbf{2}$, and hence so does $A \oplus B$. Applying the previous theorem inductively $k + 1$ times, so does $A \otimes B[\perp]^k = A \oplus B[\varobar][\perp]^k$. □

Tensor products

We can apply this in particular to the tensor product.

Corollary

If A and B are not K-S, then neither is $A \otimes B[\perp]^k$.

Proof.

If A and B are not K-S, they have homomorphisms to **2**, and hence so does $A \oplus B$. Applying the previous theorem inductively $k+1$ times, so does $A \otimes B[\perp]^k = A \oplus B[\oplus][\perp]^k$. □

Under the conjecture that $A[\perp]^*$ coincides with iterating $A[\perp]$ to a fixpoint, this would show that the logical exclusivity tensor product $A \boxtimes B$ never induces a K-S paradox if none was present if A or B.

Tensor products

We can apply this in particular to the tensor product.

Corollary

If A and B are not K-S, then neither is $A \otimes B[\perp]^k$.

Proof.

If A and B are not K-S, they have homomorphisms to $\mathbf{2}$, and hence so does $A \oplus B$. Applying the previous theorem inductively $k+1$ times, so does $A \otimes B[\perp]^k = A \oplus B[\oplus][\perp]^k$. $\square$

Under the conjecture that $A[\perp]^*$ coincides with iterating $A[\perp]$ to a fixpoint, this would show that the logical exclusivity tensor product $A \boxtimes B$ never induces a K-S paradox if none was present if A or B.

So we have narrowed, but not closed the gap ...

Duality for partial Boolean Algebras?

Our aim is to get a duality theory for pBA’s.

Duality for partial Boolean Algebras?

Our aim is to get a duality theory for pBA's.

At first sight, this looks hopeless:

- classical Stone duality for boolean algebras B builds the Stone space of B from the **points**, i.e. homomorphisms $B \longrightarrow \mathbf{2}$
- by Kochen-Specker, for interesting cases of pBA's, there are no points!

Duality for partial Boolean Algebras?

Our aim is to get a duality theory for pBA's.

At first sight, this looks hopeless:

- classical Stone duality for boolean algebras B builds the Stone space of B from the **points**, i.e. homomorphisms $B \longrightarrow \mathbf{2}$
- by Kochen-Specker, for interesting cases of pBA's, there are no points!

We will instead generalize the **Tarski duality** for complete atomic Boolean algebras (CABAs)

CABAs

Definition (Complete Boolean algebra)

A Boolean algebra A is said to be **complete** if any subset of elements $S \subseteq A$ has a supremum $\bigvee S$ in A (and consequently an infimum $\bigwedge S$, too). It thus has additional operations

$$\bigwedge, \bigvee : \mathcal{P}(A) \longrightarrow A \ .$$

Definition (Atomic Boolean algebra)

An **atom** of a Boolean algebra is a minimal non-zero element, i.e. an element $x \neq 0$ such that $a \leq x$ implies $a = 0$ or $a = x$.

Atoms are “state descriptions” or “possible worlds”.

A Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **CABA** is a complete, atomic Boolean algebra.

Tarski duality

$$\begin{tikzcd} \mathbf{CABA} \arrow[rr, bend right, "\mathrm{At}"'] \arrow[rr, phantom, "\cong"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{P}"'] \end{tikzcd}$$

Tarski duality

$$\begin{tikzcd} \mathbf{CABA} \arrow[rr, bend right, "\mathrm{At}"'] \arrow[rr, phantom, "\cong"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{P}"'] \end{tikzcd}$$

$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CABA}$ is the contravariant powerset functor:

- on objects: a set X is mapped to its powerset $\mathcal{P}X$ (a CABA).
- on morphisms: a function $f : X \longrightarrow Y$ yields a complete Boolean algebra homomorphism

$$\begin{aligned} \mathcal{P}(f) : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X) \\ (T \subseteq Y) &\longmapsto f^{-1}(T) = \{x \in X \mid f(x) \in T\} \end{aligned}$$

Tarski duality

$$\begin{tikzcd} \mathbf{CABA} \arrow[rr, bend right, "\mathrm{At}"'] \arrow[rr, phantom, "\cong"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{P}"'] \end{tikzcd}$$

$\mathbf{At} : \mathbf{CABA}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ is defined as follows:

- on objects: a CABA A is mapped to its set of atoms.
- on morphisms: a complete Boolean homomorphism $h : A \longrightarrow B$ yields a function

$$\mathrm{At}(h) : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$$

mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

Tarski duality

$$\begin{tikzcd} \mathbf{CABA} \arrow[rr, bend right, "\mathsf{At}"'] \arrow[rr, phantom, "\cong"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{P}"'] \end{tikzcd}$$

$\mathbf{At} : \mathbf{CABA}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ is defined as follows:

- on objects: a CABA A is mapped to its set of atoms.
- on morphisms: a complete Boolean homomorphism $h : A \longrightarrow B$ yields a function

$$\mathsf{At}(h) : \mathsf{At}(B) \longrightarrow \mathsf{At}(A)$$

mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

Duality for partial CABAs

Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

Definition (Atomic Boolean algebra)

A partial Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

Definition (Atomic Boolean algebra)

A partial Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **partial CABA** is a complete, atomic partial Boolean algebra.

Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

Definition (Atomic Boolean algebra)

A partial Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **partial CABA** is a complete, atomic partial Boolean algebra.

Note that $\mathrm{P}(\mathcal{H})$ is a partial CABA. Atoms are the rank-1 projectors (one-dimensional subspaces), i.e. the **pure states**.

Duality for partial CABAs: the idea

Duality for partial CABAs: the idea

- The key idea is to replace **sets** by certain **graphs**.

Duality for partial CABAs: the idea

- The key idea is to replace **sets** by certain **graphs**.
- Adjacency generalizes $\neq$, thus sets embed as **complete graphs**.

Duality for partial CABAs: the idea

- The key idea is to replace **sets** by certain **graphs**.
- Adjacency generalizes $\neq$, thus sets embed as **complete graphs**.
- These exclusivity graphs are the “non-commutative spaces” in this duality.

Duality for partial CABAs: the idea

- The key idea is to replace **sets** by certain **graphs**.
- Adjacency generalizes $\neq$, thus sets embed as **complete graphs**.
- These exclusivity graphs are the "non-commutative spaces" in this duality.
- Morphism of graphs are certain relations, generalizing the functional relations which appear in classical Tarski duality.

Graph theory notions

Definition

A **graph** $(X, \#)$ is a set equipped with a symmetric irreflexive relation.

Elements of X are called vertices, while unordered pairs $\{x, y\}$ with $x \# y$ are called edges.

Graph theory notions

Definition

A **graph** $(X, \#)$ is a set equipped with a symmetric irreflexive relation.

Elements of X are called vertices, while unordered pairs $\{x, y\}$ with $x \# y$ are called edges.

Given a vertex $x \in X$ and sets of vertices $S, T \subset X$, we write:

- $x \# S$ when for all $y \in S$, $x \# y$;
- $S \# T$ when for all $x \in S$ and $y \in T$, $x \# y$;
- $x^\# := \{y \in X \mid y \# x\}$ for the neighbourhood of the vertex x;
- $S^\# := \bigcap_{x \in S} x^\# = \{y \in X \mid y \# S\}$ for the common neighbourhood of the set S.

Graph theory notions

Definition

A **graph** $(X, \#)$ is a set equipped with a symmetric irreflexive relation.

Elements of X are called vertices, while unordered pairs $\{x, y\}$ with $x \# y$ are called edges.

Given a vertex $x \in X$ and sets of vertices $S, T \subset X$, we write:

- $x \# S$ when for all $y \in S$, $x \# y$;
- $S \# T$ when for all $x \in S$ and $y \in T$, $x \# y$;
- $x^{\#} := \{y \in X \mid y\#x\}$ for the neighbourhood of the vertex x;
- $S^{\#} := \bigcap_{x\in S} x^{\#} = \{y \in X \mid y \# S\}$ for the common neighbourhood of the set S.

A **clique** is a set of pairwise-adjacent vertices, i.e. a set $K \subset X$ with $x \# K \setminus \{x\}$ for all $x \in K$.

A graph $(X, \#)$ has **finite clique cardinal** if all cliques are finite sets.

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

- $\mathrm{At}(A)$ is the set of atomic events with an exclusivity relation.
- Can interpret these as *worlds of maximal information* and incompatibility between them.

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

- $\mathrm{At}(A)$ is the set of atomic events with an exclusivity relation.
- Can interpret these as *worlds of maximal information* and incompatibility between them.
- If A is a Boolean algebra, then $\mathrm{At}(A)$ is the complete graph on the set of atoms ($\#$ is $\neq$).

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

- $\mathrm{At}(A)$ is the set of atomic events with an exclusivity relation.
- Can interpret these as *worlds of maximal information* and incompatibility between them.
- If A is a Boolean algebra, then $\mathrm{At}(A)$ is the complete graph on the set of atoms ($\#$ is $\neq$).

Recall that in a CABA, any element is uniquely written as a join of atoms, viz. $a = \bigvee U_a$ with

$$U_a := \{x \in \mathrm{At}(A) \mid x \leq a\}$$

In a pBA, U_a may not be pairwise commeasurable, hence their join need not even be defined.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

So an element a is the join of **any** clique that is maximal in U_a.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

So an element a is the join of **any** clique that is maximal in U_a.

Given two maximal cliques K and L, this yields an equality

$$\bigvee K = \bigvee L$$

where the elements in $\bigvee K$ and those in $\bigvee L$ are not commeasurable.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

So an element a is the join of **any** clique that is maximal in U_a.

Given two maximal cliques K and L, this yields an equality

$$\bigvee K = \bigvee L$$

where the elements in $\bigvee K$ and those in $\bigvee L$ are not commeasurable.

The key to reconstructing a partial CABA from its atoms lies in characterising such equalities,

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

So an element a is the join of **any** clique that is maximal in U_a.

Given two maximal cliques K and L, this yields an equality

$$\bigvee K = \bigvee L$$

where the elements in $\bigvee K$ and those in $\bigvee L$ are not commeasurable.

The key to reconstructing a partial CABA from its atoms lies in characterising such equalities,

Proposition

Let K and L be cliques in $\mathrm{At}(A)$. Then $\bigvee K = \bigvee L$ iff $K^{\#} = L^{\#}$.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \quad :\Leftrightarrow \quad K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \quad :\Leftrightarrow \quad K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

Partial CABA from its graph of atoms

Writing

$$K \equiv L \;:\Leftrightarrow\; K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $L \sqcup K$ is a maximal clique.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \;:\Leftrightarrow\; K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $L \sqcup K$ is a maximal clique.
- $[K] \odot [L]$ iff there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \quad :\Leftrightarrow \quad K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $L \sqcup K$ is a maximal clique.
- $[K] \odot [L]$ iff there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.
- $[K] \vee [L] = [K' \cup L']$.
- $[K] \wedge [L] = [K' \cap L']$.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $L \sqcup K$ is a maximal clique.
- $[K] \odot [L]$ iff there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.
- $[K] \vee [L] = [K' \cup L']$.
- $[K] \wedge [L] = [K' \cap L']$.

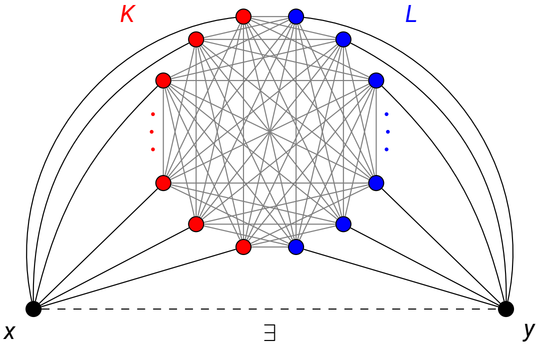
$\boxed{\text{Which conditions on a graph } (X, \#) \text{ allow for such reconstruction?}}$

Complete exclusivity graphs

Definition

A **complete exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^{\#} \# L^{\#}$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
2. $x^{\#} \subseteq y^{\#}$ implies $x = y$.



Complete exclusivity graphs

Definition

A **complete exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^{\#} \# L^{\#}$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
2. $x^{\#} \subseteq y^{\#}$ implies $x = y$.

A helpful intuition is to see these as generalising sets with a $\neq$ relation (the complete graph).

- A graph is symmetric and irreflexive.
- To be an inequivalence relation, we need cotransitivity: $x \# z$ implies $x \# y$ or $y \# z$.

Complete exclusivity graphs

Definition

A **complete exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^{\#} \# L^{\#}$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
2. $x^{\#} \subseteq y^{\#}$ implies $x = y$.

A helpful intuition is to see these as generalising sets with a $\neq$ relation (the complete graph).

- A graph is symmetric and irreflexive.
- To be an inequivalence relation, we need cotransitivity: $x \# z$ implies $x \# y$ or $y \# z$.
- Condition 1. is a weaker version of cotransitivity.
- Condition 2. eliminates redundant elements: cotransitive + 2. implies $\neq$.

Graph of atoms is complete exclusivity graph

Proposition

Let A be a partial Boolean algebra. Then $\mathrm{At}(A)$ is a complete exclusivity graph.

Proof.

Let $K, L \subset X$ such that $K \sqcup L$ is a maximal clique, and let x, y be atoms of A.
$c := \bigvee K = \neg \bigvee L$.
$x \# K$ means $x \leq \neg \bigvee K = \neg c$ and $x \# L$ means $y \leq \neg \bigvee L = c$.
By transitivity, we conclude that $x \odot y$, $\square$

Graph of atoms is complete exclusivity graph

Proposition

Let A be a partial Boolean algebra. Then $\mathrm{At}(A)$ is a complete exclusivity graph.

Proof.

Let $K, L \subset X$ such that $K \sqcup L$ is a maximal clique, and let x, y be atoms of A.
$c := \bigvee K = \neg \bigvee L$.
$x \# K$ means $x \leq \neg \bigvee K = \neg c$ and $x \# L$ means $y \leq \neg \bigvee L = c$.
By transitivity, we conclude that $x \odot y$, $\square$

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x\ R\ y$, $x'\ R\ y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

Morphisms of complete exclusivity graphs

$\boxed{\text{What about morphisms?}}$

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x\ R\ y, x'\ R\ y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. $xRy, x'Ry'$, and $y \neq y'$ implies $x \neq x'$.

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \; R \; y, x' \; R \; y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. $xRy, x'Ry'$, and $y \neq y'$ implies $x \neq x'$.($x = x'$ implies $y = y'$. (functional))

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \; R \; y, x' \; R \; y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. $xRy, x'Ry'$, and $y \neq y'$ implies $x \neq x'$.
2. $R^{-1}(Y) = X$. (left-total)

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x\ R\ y, x'\ R\ y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. $xRy, x'Ry'$, and $y \neq y'$ implies $x \neq x'$.
2. $R^{-1}(Y) = X$. (left-total)

Morphisms of complete exclusivity graphs

$\boxed{\text{What about morphisms?}}$

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \; R \; y, x' \; R \; y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. $xRy, x'Ry'$, and $y \neq y'$ implies $x \neq x'$.
2. $R^{-1}(Y) = X$. (left-total)
3. trivialises.

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \; R \; y, x' \; R \; y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. $xRy, x'Ry'$, and $y \neq y'$ implies $x \neq x'$.
2. $R^{-1}(Y) = X$. (left-total)
3. trivialises.

Given $h : A \longrightarrow B$ define $y \; R \; x$ iff $y \leq h(x)$.

Morphisms of CE graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

$$xR_hy \qquad \textit{iff} \qquad x \leq h(y)$$

is a morphism of complete exclusivity graphs. Moreover, the assignment $h \mapsto R_h$ is functorial.

Morphisms of CE graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

$$xR_hy \qquad \textit{iff} \qquad x \leq h(y)$$

is a morphism of complete exclusivity graphs. Moreover, the assignment $h \mapsto R_h$ is functorial.

Proposition

Let X and Y be complete exclusivity graphs. Given $R : X \longrightarrow Y$ a morphism of complete exclusivity graphs, the function $h_R : \mathcal{K}(Y) \longrightarrow \mathcal{K}(X)$ given by $h_R([K]) := [L]$ where L is any clique maximal in $R^{-1}(K)$ is a well-defined partial CABA homomorphism.

Morphisms of CE graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

$$xR_hy \qquad \textit{iff} \qquad x \leq h(y)$$

is a morphism of complete exclusivity graphs. Moreover, the assignment $h \mapsto R_h$ is functorial.

Proposition

Let X and Y be complete exclusivity graphs. Given $R : X \longrightarrow Y$ a morphism of complete exclusivity graphs, the function $h_R : \mathcal{K}(Y) \longrightarrow \mathcal{K}(X)$ given by $h_R([K]) := [L]$ where L is any clique maximal in $R^{-1}(K)$ is a well-defined partial CABA homomorphism.

Proposition

For any A and B be transitive partial CABAs, $\mathbf{epCABA}(A, B) \cong \mathbf{XGph}(\mathrm{At}(B), \mathrm{At}(A))$.

Global points

Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

Global points

Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

i.e. a subset of atoms of A satisfying:

1. it is an independent (or stable) set
2. it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

Global points

Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

i.e. a subset of atoms of A satisfying:

1. it is an independent (or stable) set
2. it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

The extensive literature on Kochen-Specker constructions is concerned with building graphs which have no such transversals, thus showing that the corresponding pBA's have no points.

Free-forgetful adjunction for CABAs

$$
\begin{tikzcd}
\mathbf{CABA} \arrow[rr, shift left, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{P}"] \arrow[dl, shift left, "\mathcal{P}"] \\
& \mathbf{Set} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\top"] \arrow[ur, shift left, "\mathcal{P}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}
$$

Free-forgetful adjunction for CABAs

$$
\begin{tikzcd}
\mathbf{CABA} \arrow[rr, shift left=2, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left=2, "U"] \arrow[dr, phantom, "\dashv"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, shift left=2, "\mathcal{P}"] \arrow[dl, shift left=2, "\mathcal{P}"] \arrow[dl, phantom, "\dashv"] \\
& \mathbf{Set} \arrow[ul, shift left=2, "F"] \arrow[ur, shift left=2, "\mathcal{P}"] &
\end{tikzcd}
$$

- Under the duality, it corresponds to the contravariant powerset self-adjunction.
- It gives the construction of the free CABA as a double powerset.

Free-forgetful adjunction for partial CABAs

$$\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{RGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}$$

Free-forgetful adjunction for partial CABAs

$$\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{RGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}$$

- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$

Free-forgetful adjunction for partial CABAs

$$\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{RGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\vdash"] &
\end{tikzcd}$$

- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$

- Under duality it corresponds to adjunction between **compatibility** and **exclusivity** graphs.
- This gives a concrete construction of the free CABA.

Free-forgetful adjunction for partial CABAs

$$
\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left=2, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left=2, "U"] \arrow[dr, phantom, "\perp"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left=2, "\mathcal{K}"] \arrow[dl, shift left=2, red, "\mathcal{K}"] \arrow[dl, phantom, "\perp"] \\
& \mathbf{RGph} \arrow[ul, shift left=2, "F"] \arrow[ur, shift left=2, red, "\mathcal{V}"] &
\end{tikzcd}
$$

- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$
- Under duality it corresponds to adjunction between **compatibility** and **exclusivity** graphs.
- This gives a concrete construction of the free CABA. A compatibility $\langle P, \odot \rangle$ to a graph with vertices $\langle C, \gamma : C \longrightarrow \{0,1\} \rangle$ where C maximal compatible set, and edges

$$\langle C, \gamma \rangle \# \langle D, \delta \rangle \qquad \text{iff} \qquad \exists x \in C \cap D.\; \gamma(x) \neq \delta(x)\,.$$

Kochen-Specker paradoxes and Mermin squares

We recall the following result:

Theorem

Let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to* **2***)*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$*,*

$$A \models \varphi(\vec{a})$$

Kochen-Specker paradoxes and Mermin squares

We recall the following result:

Theorem

Let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to **2**)*
2. *For some **propositional contradiction** $\varphi(\vec{x})$ and assignment $\vec{x} \mapsto \vec{a}$,*

$$A \models \varphi(\vec{a})$$

We want to explicitly construct such a contradiction which evaluates to true in $\mathrm{P}(\mathcal{H})$.

Kochen-Specker paradoxes and Mermin squares

We recall the following result:

Theorem

Let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to* **2***)*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$*,*

$$A \models \varphi(\vec{a})$$

We want to explicitly construct such a contradiction which evaluates to true in $\mathrm{P}(\mathcal{H})$.

While we can do this by encoding colouring problems on sets of vectors, there is a more elegant approach which yields a smaller formula.

Kochen-Specker paradoxes and Mermin squares

We recall the following result:

Theorem

Let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to* **2***)*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$*,*

$$A \models \varphi(\vec{a})$$

We want to explicitly construct such a contradiction which evaluates to true in $\mathrm{P}(\mathcal{H})$.

While we can do this by encoding colouring problems on sets of vectors, there is a more elegant approach which yields a smaller formula.

This also provides an opportunity to make contact with another important idea, the **Pauli group**.

The Pauli group on qubits

We recall the definition of the **Pauli operators** on $\mathbb{C}^2$, dichotomic (i.e. two-valued) observables corresponding to measuring spin in the *x*, *y*, and *z* axes, with eigenvalues ± 1

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices are self-adjoint, have eigenvalues ± 1, and together with the identity matrix I satisfy the following relations:

$$\begin{gathered} X^2 = Y^2 = Z^2 = I \\ XY = iZ, \;\; YZ = iX, \;\; ZX = iY, \\ YX = -iZ, \;\; ZY = -iX, \;\; XZ = -iY. \end{gathered} \tag{1}$$

The Pauli 2-group

We can extend this to a group operating on 2 qubits, $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The Pauli 2-group

We can extend this to a group operating on 2 qubits, $\mathbb{C}^2 \otimes \mathbb{C}^2$.

We write $XI := X \otimes I$, etc.

The Pauli 2-group

We can extend this to a group operating on 2 qubits, $\mathbb{C}^2 \otimes \mathbb{C}^2$.

We write $XI := X \otimes I$, etc.

By bilinearity of tensor, we have

$$\alpha U \otimes \beta V = \alpha\beta(U \otimes V)$$

The Pauli 2-group

We can extend this to a group operating on 2 qubits, $\mathbb{C}^2 \otimes \mathbb{C}^2$.

We write $XI := X \otimes I$, etc.

By bilinearity of tensor, we have

$$\alpha U \otimes \beta V = \alpha\beta(U \otimes V)$$

Thus e.g. we have

$$(XZ)(ZX) = (-i)iYY = YY$$

while

$$(XX)(ZZ) = i^2 YY = -YY$$

The Peres-Mermin magic square

Now we can define a famous and important construction, the **Peres-Mermin magic square**:

$$
\begin{array}{ccccc}
XI & - & IX & - & XX \\
| & & | & & | \\
IZ & - & ZI & - & ZZ \\
| & & | & & | \\
XZ & - & ZX & - & YY
\end{array}
$$

Note that:

- The operators in each row and column commute.
- The product of each of the rows, and of the first two columns, is II.
- The product of the third column is $-II$.

Contextuality in the P-M square

We ask if there is a **non-contextual value assignment** val : $\mathcal{X} \longrightarrow \mathbb{Z}_2$, where $\mathcal{X}$ is the set of operators in the table, subject to the conditions that

1. if p and q commute, then $\mathrm{val}(pq) = \mathrm{val}(p) + \mathrm{val}(q)$.
2. $\mathrm{val}(II) = 0$ and $\mathrm{val}(-II) = 1$.

Contextuality in the P-M square

We ask if there is a **non-contextual value assignment** val : $\mathcal{X} \longrightarrow \mathbb{Z}_2$, where $\mathcal{X}$ is the set of operators in the table, subject to the conditions that

1. if p and q commute, then $\mathrm{val}(pq) = \mathrm{val}(p) + \mathrm{val}(q)$.
2. $\mathrm{val}(II) = 0$ and $\mathrm{val}(-II) = 1$.

If there were such an assignment, we would have a solution for the following set of equations over $\mathbb{Z}_2$ from the above table, one for each row and each column:

$$\begin{array}{rcl@{\qquad\qquad}rcl} a+b+c & = & 0 & a+d+g & = & 0 \\ d+e+f & = & 0 & b+e+h & = & 0 \\ g+h+i & = & 0 & c+f+i & = & 1 \end{array} \tag{2}$$

Here a is a variable corresponding to $\mathrm{val}(XI)$, etc.

Contextuality in the P-M square

We ask if there is a **non-contextual value assignment** val : $\mathcal{X} \longrightarrow \mathbb{Z}_2$, where $\mathcal{X}$ is the set of operators in the table, subject to the conditions that

1. if p and q commute, then $\mathrm{val}(pq) = \mathrm{val}(p) + \mathrm{val}(q)$.
2. $\mathrm{val}(II) = 0$ and $\mathrm{val}(-II) = 1$.

If there were such an assignment, we would have a solution for the following set of equations over $\mathbb{Z}_2$ from the above table, one for each row and each column:

$$\begin{array}{rclcrcl} a+b+c & = & 0 & \qquad & a+d+g & = & 0 \\ d+e+f & = & 0 & & b+e+h & = & 0 \\ g+h+i & = & 0 & & c+f+i & = & 1 \end{array} \tag{2}$$

Here a is a variable corresponding to $\mathrm{val}(XI)$, etc.

Summing the left hand sides yields 0, summing the right hand sides yields 1, contradiction.

The partial homomorphism condition

The justification for assuming the partial homomorphism condition comes from the quantum case:

- if A and B are commuting observables and ψ is a common eigenvector of A and B, with eigenvalue v for A and w for B, then ψ is an eigenvector for AB with eigenvalue vw.

[1] Note that $\{+1, -1\}$ under multiplication is an isomorphic representation of $\mathbb{Z}_2$, with 0 corresponding to $+1$ and 1 to -1 under the mapping $i \mapsto (-1)^i$.

The partial homomorphism condition

The justification for assuming the partial homomorphism condition comes from the quantum case:

- if A and B are commuting observables and ψ is a common eigenvector of A and B, with eigenvalue v for A and w for B, then ψ is an eigenvector for AB with eigenvalue vw.

Also, $I\!I$ has the unique eigenvalue $+1$, and $-I\!I$ the unique eigenvalue -1.[1]

[1] Note that $\{+1, -1\}$ under multiplication is an isomorphic representation of $\mathbb{Z}_2$, with 0 corresponding to $+1$ and 1 to -1 under the mapping $i \mapsto (-1)^i$.

The partial homomorphism condition

The justification for assuming the partial homomorphism condition comes from the quantum case:

- if A and B are commuting observables and ψ is a common eigenvector of A and B, with eigenvalue v for A and w for B, then ψ is an eigenvector for AB with eigenvalue vw.

Also, $I\!I$ has the unique eigenvalue $+1$, and $-I\!I$ the unique eigenvalue -1.[1]

This is Kochen and Specker's refinement of von Neumann's much criticized no-go theorem.

[1] Note that $\{+1, -1\}$ under multiplication is an isomorphic representation of $\mathbb{Z}_2$, with 0 corresponding to $+1$ and 1 to -1 under the mapping $i \mapsto (-1)^i$.

From Paulis to projectors

Theorem

There is a bijective correspondence between unitary involutions u (i.e. $u = u^*$, $u^2 = I$) and projectors p, given by

- $u = 2p - I$
- $p = \frac{1}{2}(I + u)$

Moreover, the correspondence preserves and reflects commutation of products, and

- *if p corresponds to u, then $I - p$ corresponds to $-u$*
- *If p corresponds to u and q to v, and p commutes with q, then $p \leftrightarrow q$ corresponds to uv.*

Here in a pBA, if a is compatible with b, then $a \leftrightarrow b := (a \wedge b) \vee (\neg a \wedge \neg b)$.

From Paulis to projectors

Theorem

There is a bijective correspondence between unitary involutions u (i.e. $u = u^*$, $u^2 = I$) and projectors p, given by

- $u = 2p - I$
- $p = \frac{1}{2}(I + u)$

Moreover, the correspondence preserves and reflects commutation of products, and

- *if p corresponds to u, then $I - p$ corresponds to $-u$*
- *If p corresponds to u and q to v, and p commutes with q, then $p \leftrightarrow q$ corresponds to uv.*

Here in a pBA, if a is compatible with b, then $a \leftrightarrow b := (a \wedge b) \vee (\neg a \wedge \neg b)$.

Thus we can translate **algebraic paradoxes** in the Paulis into **logical paradoxes** in the pBA of projectors.

Contextual words

A **contextual word** in the Pauli 2-group is a product

$$w = x_1 \cdots x_n$$

such that:

- w can be built up from commuting products
- each element occurs in w an even number of times
- $w = -II$.

A contextual word is a witness for contextuality, since it shows that no non-contextual value assignment can exist.

Contextual words

A **contextual word** in the Pauli 2-group is a product

$$w = x_1 \cdots x_n$$

such that:

- w can be built up from commuting products
- each element occurs in w an even number of times
- $w = -II$.

A contextual word is a witness for contextuality, since it shows that no non-contextual value assignment can exist.

A contextual word corresponding to the Peres-Mermin square is

$$((XI\,IZ)(ZI\,IX))((XI\,IX)(ZI\,IZ))$$

Note that first principal subterm evaluates to $XZ\,ZX = YY$, the second to $XX\,ZZ = -YY$.

From contextual words to paradoxes

We can now use the correspondence between involutive unitaries and projectors to turn the contextual word into a tautology falsified in the pBA of projectors.

From contextual words to paradoxes

We can now use the correspondence between involutive unitaries and projectors to turn the contextual word into a tautology falsified in the pBA of projectors.

We have the projectors corresponding to the four local Paulis used to construct the contextual word:

$$a = p(XI), \quad b = p(IZ), \quad c = p(ZI), \quad d = p(IX)$$

From contextual words to paradoxes

We can now use the correspondence between involutive unitaries and projectors to turn the contextual word into a tautology falsified in the pBA of projectors.

We have the projectors corresponding to the four local Paulis used to construct the contextual word:

$$a = p(XI), \quad b = p(IZ), \quad c = p(ZI), \quad d = p(IX)$$

We can turn the contextual word

$$((XI\,IZ)(ZI\,IX))((XI\,IX)(ZI\,IZ))$$

into the classical tautology

$$([a \leftrightarrow b] \leftrightarrow [c \leftrightarrow d]) \leftrightarrow ([a \leftrightarrow d] \leftrightarrow [c \leftrightarrow b])$$

From contextual words to paradoxes

We can now use the correspondence between involutive unitaries and projectors to turn the contextual word into a tautology falsified in the pBA of projectors.

We have the projectors corresponding to the four local Paulis used to construct the contextual word:

$$a = p(XI), \quad b = p(IZ), \quad c = p(ZI), \quad d = p(IX)$$

We can turn the contextual word

$$((XI\,IZ)(ZI\,IX))((XI\,IX)(ZI\,IZ))$$

into the classical tautology

$$([a \leftrightarrow b] \leftrightarrow [c \leftrightarrow d]) \leftrightarrow ([a \leftrightarrow d] \leftrightarrow [c \leftrightarrow b])$$

The fact that the word evaluates to $-II$ means that this tautology evaluates to false.

From contextual words to paradoxes

We can now use the correspondence between involutive unitaries and projectors to turn the contextual word into a tautology falsified in the pBA of projectors.

We have the projectors corresponding to the four local Paulis used to construct the contextual word:

$$a = p(XI), \quad b = p(IZ), \quad c = p(ZI), \quad d = p(IX)$$

We can turn the contextual word

$$((XI\,IZ)(ZI\,IX))((XI\,IX)(ZI\,IZ))$$

into the classical tautology

$$([a \leftrightarrow b] \leftrightarrow [c \leftrightarrow d]) \leftrightarrow ([a \leftrightarrow d] \leftrightarrow [c \leftrightarrow b])$$

The fact that the word evaluates to $-II$ means that this tautology evaluates to false.

Similarly, the classical contradiction

$$([a \leftrightarrow b] \leftrightarrow [c \leftrightarrow d]) \oplus ([a \leftrightarrow d] \leftrightarrow [c \leftrightarrow b])$$

evaluates to true.
Here $e \oplus f := (e \wedge \neg f) \vee (\neg e \wedge f)$.

Further notes

A result by Coray shows that every 3-variable classical tautology is satisfied in pBA's. Thus this 4-variable example is minimal.

Further notes

A result by Coray shows that every 3-variable classical tautology is satisfied in pBA's. Thus this 4-variable example is minimal.

However, the classical contradiction

$$(a \leftrightarrow b) \wedge (b \leftrightarrow c) \wedge (c \leftrightarrow d) \wedge (d \oplus a)$$

corresponding to the CHSH game/PR box is **not** satisfiable in any transitive pBA.

Further notes

A result by Coray shows that every 3-variable classical tautology is satisfied in pBA's. Thus this 4-variable example is minimal.

However, the classical contradiction

$$(a \leftrightarrow b) \wedge (b \leftrightarrow c) \wedge (c \leftrightarrow d) \wedge (d \oplus a)$$

corresponding to the CHSH game/PR box is **not** satisfiable in any transitive pBA.

We can consider the following question:

- Given a classical contradiction φ, is this satisfied in a projection lattice?

Question If the dimension is unbounded is this decidable? if we bound the dimension, what is the complexity?
We can ask similar questions for satisfiability in classes of pBA's.

Further notes

A result by Coray shows that every 3-variable classical tautology is satisfied in pBA's. Thus this 4-variable example is minimal.

However, the classical contradiction

$$(a \leftrightarrow b) \wedge (b \leftrightarrow c) \wedge (c \leftrightarrow d) \wedge (d \oplus a)$$

corresponding to the CHSH game/PR box is **not** satisfiable in any transitive pBA.

We can consider the following question:

- Given a classical contradiction φ, is this satisfied in a projection lattice?

Question If the dimension is unbounded is this decidable? if we bound the dimension, what is the complexity?
We can ask similar questions for satisfiability in classes of pBA's.

We can also generalize beyond the Pauli group considered here. See SA, Carmen Constatin and Serban Cercelescu, *Commutation Groups and state-independent contextuality*, to appear at FSCD 2024, also presentation at TACL.

Further notes

A result by Coray shows that every 3-variable classical tautology is satisfied in pBA's. Thus this 4-variable example is minimal.

However, the classical contradiction

$$(a \leftrightarrow b) \wedge (b \leftrightarrow c) \wedge (c \leftrightarrow d) \wedge (d \oplus a)$$

corresponding to the CHSH game/PR box is **not** satisfiable in any transitive pBA.

We can consider the following question:

- Given a classical contradiction φ, is this satisfied in a projection lattice?

Question If the dimension is unbounded is this decidable? if we bound the dimension, what is the complexity?
We can ask similar questions for satisfiability in classes of pBA's.

We can also generalize beyond the Pauli group considered here. See SA, Carmen Constatin and Serban Cercelescu, *Commutation Groups and state-independent contextuality*, to appear at FSCD 2024, also presentation at TACL.
Does the connection to logic and pBA's persist in these generalizations?