A categorical approach to automata learning and minimization – part 1

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TACL'24, Barcelona, 24-28 June 2024

INSTITUT
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EN INFORMATIQUE
FONDAMENTALE

Université de Paris

TANCL'07 in Oxford

- the first conference I attended
- organized by Mai Gehrke and Hilary Priestley
- some wonderful talks Samson Abramsky, Alexander Kurz, Jean-Éric Pin, etc.
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References for Lecture 1

T. Colcombet and D. Petrişan. *Automata minimization: a functorial approach*. Log. Methods Comput. Sci., 16(1), 2020

J. E. Pin (Ed.) *Handbook of Automata Theory*, EMS Press, 2021

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the interplay between category theory and automata theory.

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- provides a unifying framework for modelling various forms of automata,
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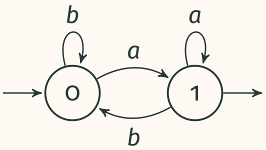
In particular, we will see how the category-theoretic approach

- provides a unifying framework for modelling various forms of automata,
- for obtaining generic algorithms for learning algorithms,
- highlights the link between automata learning and minimization.

Automata – the basics

A complete deterministic finite automaton over some finite alphabet A is a tuple $\mathcal{A} = (Q, q_0, F, (\delta_a)_{a \in A})$ where

- Q is a finite set of states
- q_0 is an element of Q, called initial state
- $F \subseteq Q$ is a subset of accepting states
- for every letter $a \in A$, $\delta_a : Q \to Q$ is a transition function



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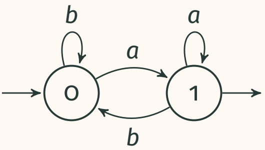

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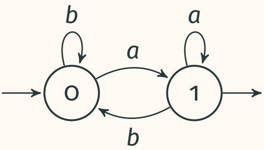

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A word $w \in A^*$ is accepted by $\mathcal{A}$ when $\delta_w(q_0) \in F$.
The language of $\mathcal{A}$ is the set $\mathcal{L}(\mathcal{A})$ of words over A^* accepted by $\mathcal{A}$.

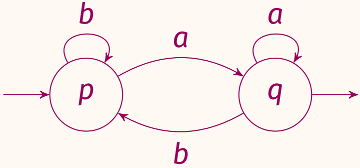
Regular languages

... form a very robust class – described in a multitude of ways.

Example: “Last letter is a.”

$(a+b)^*a$

the language of a regular expression (Kleene theorem)



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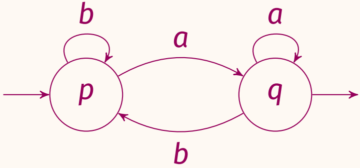

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$\phi\colon A^* \to \{1, a, b\}$

the preimage of a monoid morphism

	1	a	b
1	1	a	b
a	a	a	b
b	b	a	b

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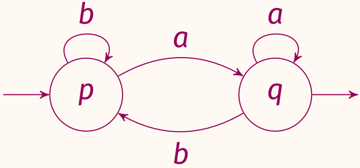

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$\exists x.\ \neg(\exists y.\ x < y) \wedge Q_a x$

definable in MSO (Büchi-Elgot-Trakhtenbrot)

Minimization

Given a language $L \subseteq A^*$ and a word $u \in A^*$ the left quotient $u^{-1}L$ is the set

$$\{v \in A^* \mid uv \in L\}$$

The Myhill-Nerode equivalence is defined by

$$u \cong_L v \text{ iff } u^{-1}L = v^{-1}L$$

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Theorem (Myhill-Nerode). A language L is regular iff it has only finitely many left quotients iff $\cong_L$ has finite index.

Proof. $\Rightarrow$ If an automaton $\mathcal{A} = (Q, q_0, F, (\delta_a)_{a \in A})$ accepts a language L, then the automaton $(Q, \delta_u(q_0), F, (\delta_a)_{a \in A})$ accepts $u^{-1}L$.

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$\Leftarrow$ Consider the Nerode automaton of L, that is $(Q, q_0, F, (\delta_a)_{a \in A})$, where

- $Q = \{u^{-1}L \mid u \in A^*\}$,
- $q_0 = L$
- $F = \{u^{-1}L \mid u \in L\}$ and
- $\delta_a(u^{-1}L) = (ua)^{-1}L$.

Minimization

How do we minimize an automaton $\mathcal{A}$?

- remove all states that are not accessible from the initial state. We obtain the reachable sub-automaton $\mathrm{Reach}(\mathcal{A})$.
- Merge all states that accept the same language, we obtain the observable quotient $\mathrm{Obs}(\mathrm{Reach}(\mathcal{A}))$.

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There are several algorithms for minimizing automata:
Moore, Hopcroft, Brzozowski.

Non-deterministic automata

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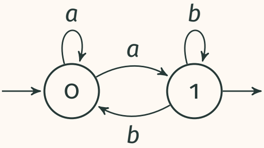

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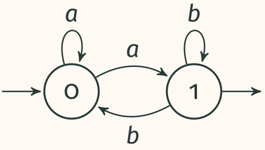

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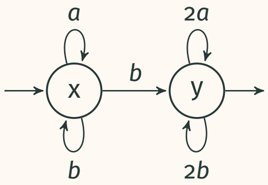
Proposition. Every NFA is equivalent to a DFA.

Weighted automata over a semiring

Given a semiring S, an S-weighted automaton over some finite alphabet $A$is a tuple $\mathcal{A} = (Q, i, f, \delta)$ where

- Q is a finite set of states
- $i\colon Q \to S$ assigns an initial value to each state
- $f\colon Q \to S$ is a subset a final value to each state
- $\delta\colon Q \times A \times Q \to S$ assigns to each transition a value in S

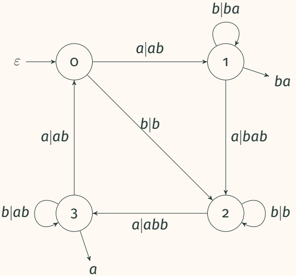


- Let $w \in A^*$. For an accepting path labelled by w compute its weight using the multiplication of the semiring.
- We add the weights of all accepting pathes labelled by w to obtain $\mathcal{L}(\mathcal{A})(w)$.

Sequential transducers

A sequential transducer with input alphabet A and output alphabet B consists of:

- a finite set of states Q
- an initial state with an initial output in B^*, or an undefined initial state



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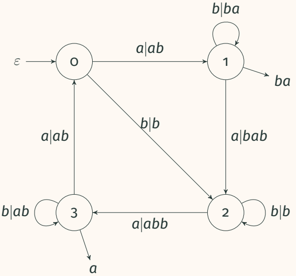

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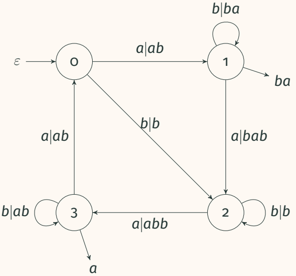
- for each state in Q, either an output word in B^* or undefined.



A unifying framework for automata minimization

Very basic notions of category theory

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- for every object A, an identity morphism $1_A\colon A \to A$
- a partial composition $\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

Additionally the composition should satisfy unit and associativity axioms.

Examples of categories

To get the gist of the remaining slides, we basically need to understand 4-5 examples of categories :

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- Set – the category of sets and functions
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- Vec – the category of vector spaces and linear transformations
- $\mathcal{T}$ – the category of free partial actions of some free monoid B^* and their morphisms
- the free category on a graph, in particular

$$\begin{tikzcd} \text{in} \arrow[r, "\triangleright"] & \text{states} \arrow[loop above, "a\ (a \in A)"] \arrow[r, "\triangleleft"] & \text{out} \end{tikzcd}$$

Many more, that you have surely encountered: (semi)groups, monoids, topological spaces, etc.

Word automata

deterministic automata $\quad 1 \longrightarrow Q \longrightarrow 2 \quad$ in Set

$$\begin{tikzcd} 1 \arrow[r] & Q \arrow[loop above, "a"] \arrow[r] & 2 \end{tikzcd}$$

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Seq. transducers $\begin{tikzcd} 1 \arrow[r] & Q \arrow[r] \arrow["a", loop above] & 1 \end{tikzcd}$ in $\mathcal{T}$

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- objects: sets $X, Y, Z, \ldots$
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Composition of arrows in $\mathcal{T}$ is defined using the monoid multiplication in B^*.

If $f : X \nrightarrow Y$ and $g : Y \nrightarrow Z$ then $g \circ f : X \nrightarrow Z$ (i.e. $g \circ f : X \to B^* \times Z + 1$) is given by $g \circ f(x) = \begin{cases} (uv, z) & \text{if } f(x) = (u, y) \text{ and } g(y) = (v, z) \\ \perp & \text{otherwise.} \end{cases}$

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This is the Kleisli category for the monad $T\colon \mathrm{Set} \to \mathrm{Set}$ given by $T(X) = B^* \times X + 1$, which associates to each set X the free partial action of B^* on X.

The output category for subsequential transducers

Interpretting the arrows

$$\begin{tikzcd} 1 \arrow[r, "i"] & Q \arrow[r, "f"] \arrow[loop above, "\delta_a"] & 1 \end{tikzcd} \qquad \text{in } \mathcal{T}$$

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- a function $i : 1 \to B^* \times Q + 1$, i.e. an initial state with an initial output in B^*, or an undefined initial state
- for each $a \in A$ a function $\delta_a : Q \to B^* \times Q + 1$
- a final map $f : Q \to B^* \times 1 + 1$, i.e. for each state in Q either an output word in B^* or undefined.

What does “interpretting” mean?

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It’s all about functors – which are to categories what functions are to sets !!

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Definition. Given categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- for each object A of $\mathcal{C}$, an object FA of $\mathcal{D}$
- for each arrow $f : A \to B$ in $\mathcal{C}$, an arrow $Ff : FA \to FB$ in $\mathcal{D}$

such that identities and composition are preserved:
$F(1_A) = 1_{FA}$ and $Ff \circ Fg = F(f \circ g)$ when $f \circ g$ is defined.

Word automata as functors

Word automata on A^* are **functors** $\mathcal{A}\colon\mathcal{I} \to \mathcal{C}$, where the input category $\mathcal{I}$ is freely generated by

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The data given by the functor $\mathcal{A}$ is a tuple $\langle Q, i, f, (\delta_a)_{a\in A}\rangle$, where

- Q is an object of $\mathcal{C}$.
- $i\colon I \to Q$ is the «initial» arrow, for some object I of $\mathcal{C}$
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The language accepted by $\mathcal{A}$ is a map $L_{\mathcal{A}}\colon A^* \to \mathcal{C}(I, F)$ that associates to a word $w = a_1 \ldots a_n$ the composite morphism

$$I \xrightarrow{i} Q \xrightarrow{\delta_{a_1}} Q \xrightarrow{\delta_{a_2}} \ldots \xrightarrow{\delta_{a_n}} Q \xrightarrow{f} F$$

Automata and languages as functors

An automaton $\mathcal{A}$ accepts a language $\mathcal{L}$ when the next diagram commutes

$$\begin{tikzcd}
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\mathrm{in} \arrow[r, "\triangleright"] & \mathrm{states} \arrow[loop above, "a"] \arrow[r, "\triangleleft"] & \mathrm{out}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O} \arrow[r, "\mathcal{L}"] \arrow[d, hook] & \mathcal{C} \\
\mathcal{I} \arrow[ur, "\mathcal{A}"'] &
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For every language $\mathcal{L}\colon \mathcal{O} \to \mathcal{C}$ we consider a category $\mathrm{Auto}_{\mathcal{L}}$ of automata accepting $\mathcal{L}$.

$\mathcal{O}$ can be seen as an “observation” subcategory of $\mathcal{I}$.

Much of the ensuing theory can be developed independently on the precise shape of $\mathcal{I}$.

Automata in a category: minimization

Minimzation of $\mathcal{C}$-automata

- What does it mean for a $\mathcal{C}$-automaton to be minimal?
- What are sufficient conditions on $\mathcal{C}$ so that a minimal automaton for a language exists?
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Thus we need a notion of «quotient» (surjection for sets) and «sub-object» (injection for sets), i.e. a factorization system.

Three more category-theoretic notions

- An initial object in a category $\mathcal{C}$ is an object X such that for any object A of $\mathcal{C}$ there is a unique morphism $! \colon X \to A$.
  Question: what is the initial object in Set*? And in* Rel*?*

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- A final object in a category $\mathcal{C}$ is an object Y such that for any object A of $\mathcal{C}$ there is a unique morphism $! \colon A \to Y$.
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- A factorization system provides the category-theoretic generalizations for the notions of “quotients” and “subobjects”, definition on next slide...

The three ingredients for minimization

When does a ‘minimal’ automaton accepting a language $\mathcal{L}$ exist?

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "\mathcal{L}"] \arrow[d, hook] & \mathcal{C} \\
\mathcal{I} \arrow[ur, dashed, "\mathtt{Min}(\mathcal{L})?"] &
\end{tikzcd}$$

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left Kan ext?

$$\begin{tikzcd} \mathcal{O} \arrow[r, "\mathcal{L}"] \arrow[d, hook] & \mathcal{C} \\ \mathcal{I} \arrow[ur, bend left, "\mathcal{A}_{\mathtt{init}}(\mathcal{L})"] & \end{tikzcd}$$

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$$
\begin{tikzcd}
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\mathcal{I} \arrow[ur, bend left=50, "\mathcal{A}_{\mathtt{init}}(\mathcal{L})"] \arrow[ur, bend right=50, "\mathcal{A}_{\mathtt{final}}(\mathcal{L})"'] &
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If the category of automata accepting $\mathcal{L}$ has

- an initial object $\mathcal{A}_{\mathtt{init}}(\mathcal{L})$, ✓ when $\mathcal{C}$ has copowers
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- a factorization system ✓ when $\mathcal{C}$ has one

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Factorization systems

Factorization systems are a generalization of the next situation: Every function $f\colon X \to Y$ can we written as a composite

$$X \xrightarrow{e} \!\!\!\!\rightarrow Z \rightarrowtail Y$$

with e a surjection and m and injection, and, moreover, such a decomposition is unique up-to isomorphism.

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- every morphism $f: X \to Y$ can we written as a composite $e \circ m$ with $e \in E$ and $m \in M$;
- the decomposition is functorial, i.e. any two decompositions are isomorphic

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Trivial example: minimizing DFAs

The initial automaton $\mathcal{A}_{\mathtt{init}}$ for Set-automata accepting a language L is the following :

$$\begin{tikzcd} 1 \arrow[r, "\varepsilon"] & A^* \arrow[loop above, "w \mapsto wa"] \arrow[r, "L?"] & 2 \end{tikzcd}$$

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The final automaton $\mathcal{A}_{\mathtt{final}}$ for Set-automata accepting a language L is the following :

$$\begin{tikzcd} 1 \arrow[r, "L"] & 2^{A^*} \arrow[loop above, "K \mapsto a^{-1}K"] \arrow[r, "\varepsilon?"] & 2 \end{tikzcd}$$

Trivial example: minimizing DFAs accepting L

The unique map from the initial to the final automaton is given by $! \colon A^* \to 2^{A^*}$, defined by $w \mapsto w^{-1}L$.

$$
\begin{tikzcd}
 & A^* \arrow[d, "\text{reachedState}"] \arrow[dr, bend left, "\mathcal{L}?"] & \\
1 \arrow[ur, bend left, "\varepsilon"] \arrow[r, "i"] \arrow[dr, bend right, "\mathcal{L}"'] & Q \arrow[r, "f"] \arrow[d, "\text{acceptedLanguage}"] & 2 \\
 & 2^{A^*} \arrow[ur, bend right, "\varepsilon?"'] &
\end{tikzcd}
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Another trivial example

$\mathbb{R}$-weighted automata, i.e. $(\mathsf{Vec}, \mathbb{R}, \mathbb{R})$-automata accepting a $(\mathsf{Vec}, \mathbb{R}, \mathbb{R})$-language

$$\begin{tikzcd}
 & \bigoplus\limits_{u \in A^*} \mathbb{R} \arrow[d, "\mathsf{reachedState}"] \arrow[dr, bend left, "L?"] & \\
\mathbb{R} \arrow[ur, bend left, "\varepsilon"] \arrow[r, "i"] \arrow[dr, bend right, "L"'] & Q \arrow[r, "F"] \arrow[d, "\mathsf{acceptedLanguage}"] & \mathbb{R} \\
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$$

The minimal transducer in a picture

We obtain $\mathtt{Min}(\mathcal{L})$ – the minimal subsequential transducer as obtained by Choffrut!

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 & \mathsf{Irr}(A^*, B^*) \arrow[ur, bend right, "\varepsilon?"'] &
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The minimal transducer in a picture

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$$
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Minimial Automaton $\mathtt{Min}(\mathcal{L})$ for a Language

The automaton $\mathtt{Min}(\mathcal{L})$ divides any other automaton accepting $\mathcal{L}$.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dr] & \\
\mathcal{A}_{\mathtt{init}}(L) \arrow[ur] & & \mathcal{A}_{\mathtt{final}}(L)
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\begin{tikzcd}
& & \mathcal{A} \arrow[drr, bend left] & & \\
\mathcal{A}_{\mathtt{init}}(L) \arrow[r, two heads] \arrow[urr, bend left] & \mathtt{reach}(\mathcal{A}) \arrow[ur, tail] & & & \mathcal{A}_{\mathtt{final}}(L)
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\end{tikzcd}
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Thus far we identified simple sufficient conditions on $\mathcal{C}$ so that minimization of $\mathcal{C}$-automata is guaranteed!

Lifting adjunctions between output categories to automata

Lifting adjunctions

Suppose we have the ‘same’ language interpretted in two different categories related by an adjunction $F \dashv U$:

$$L_C\colon A^* \to C(X, UY) \text{ and } L_D\colon A^* \to D(FX, Y).$$

$$\begin{tikzcd}
Auto(L_C) \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] \arrow[d, "States"] & & Auto(L_D) \arrow[ll, bend left] \arrow[d, "States"] \\
C \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & D \arrow[ll, bend left, "U"]
\end{tikzcd}$$

Lifting adjunctions – determinization

Suppose we have the 'same' regular language interpretted in two different categories (Set and Rel) related by an adjunction $F \dashv U$:

$$L_{\mathrm{Set}}: A^* \to \mathrm{Set}(1, U1) \text{ and } L_{\mathrm{Rel}}: A^* \to \mathrm{Rel}(F1, 1).$$

$$\begin{tikzcd}
Auto(L_{\mathrm{Set}}) \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] \arrow[d, "States"] & & Auto(L_{\mathrm{Rel}}) \arrow[ll, bend left] \arrow[d, "States"] \\
\mathrm{Set} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathrm{Rel} \arrow[ll, bend left, "U"]
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Corollary 1. The determinization of NFA is a right adjoint to inclusions of DFA in NFA.

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Corollary 1. The determinization of NFA is a right adjoint to inclusions of DFA in NFA.

Corollary 2. Initial automata for free in Kleisli valued automata.