A categorical approach to automata learning and minimization – part 3

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Automata Learning

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- The most famous learning algorithm for automata is the L^*-algorithm of Dana Angluin.
  D. Angluin, Learning Regular Sets from Queries and Counterexamples, Information and Computation, 1978

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- The algorithm stops when the teacher agrees that the hypothesis automaton accepts the language L.

The L^*-algorithm: some definitions

- At each step, we maintain a pair of sets of words (Q, T), starting with $(\{\epsilon\}, \{\epsilon\})$.
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- When (Q, T) is closed and consistent it is possible to build a hypothesis automaton $\mathcal{H}(Q, T)$

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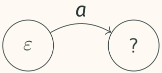

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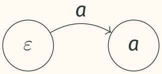

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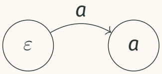

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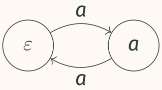

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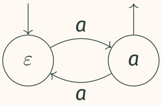

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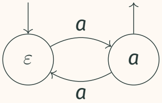

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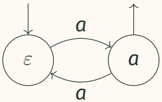

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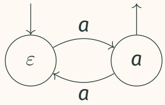

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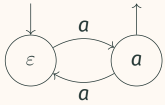

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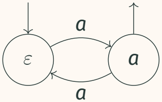
$\mathcal{H}(\{\varepsilon, a\}, \{\varepsilon\})$ 

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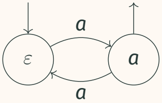

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$\varepsilon \sim_{\{\varepsilon\}} aa$, but $a \not\sim_{\{\varepsilon\}} aaa$

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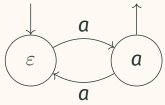
$\mathcal{H}(\{\varepsilon, a\}, \{\varepsilon\})$ 

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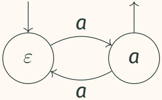

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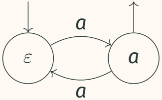

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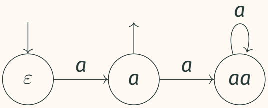

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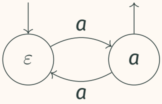

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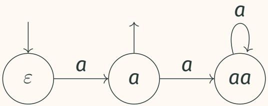

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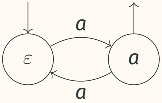
Example: Learning the language $\{a\}$ over the alphabet $\{a\}$.

Q	T
$\{\varepsilon, a, aa, aaa\}$	$\{\varepsilon, a\}$

$\mathcal{H}(\{\varepsilon, a\}, \{\varepsilon\})$



Teacher: “No! *aaa* is a counterex.”

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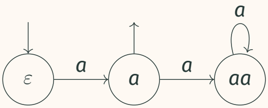

Teacher: “Yes!”

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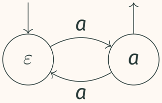

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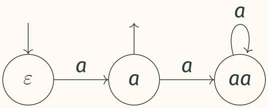

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The L^*-algorithm: Variations

The L^*-algorithm has been extended to various other forms of automata

- weighted automata over fields (Bergadano and Varricchio, 1996)
- ~~sub~~sequential transducers (Vilar, 1996)
- nominal automata (van Heerd et al., 2017)
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Other category theoretic generalizations (van Heerd et al., 2017; Urbat and Schröder, 2019)

Back to learning...
automata, not categories!

L^* algorithm categorically??

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Thomas Colcombet, Daniela Petrisan, Riccardo Stabile: Learning Automata and Transducers: A Categorical Approach. CSL 2021

L^*-revisited

- At the (Q, T) stage of the algorithm the learner only has access to a fragment of the language:

$$L_{Q,T} : QAT \cup QT \rightarrowtail A^* \xrightarrow{L} 2$$

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- This can be represented by a notion of (Q, T)-biautomaton

$$1 \xrightarrow[(q \in Q)]{\triangleright q} Q_1 \underset{\varepsilon}{\overset{a\ (a \in A)}{\rightrightarrows}} Q_2 \xrightarrow[(t \in T)]{t \triangleleft} 2$$

such that the following coherence diagrams commute

$$\begin{array}{ccccc} & & Q_1 & & \\ & \overset{\triangleright q}{\nearrow} & & \overset{a}{\searrow} & \\ 1 & & & & Q_2 \\ & \underset{\triangleright qa}{\searrow} & & \underset{\varepsilon}{\nearrow} & \\ & & Q_1 & & \end{array} \qquad \begin{array}{ccccc} & & Q_2 & & \\ & \overset{a}{\nearrow} & & \overset{t \triangleleft}{\searrow} & \\ Q_1 & & & & 2 \\ & \underset{\varepsilon}{\searrow} & & \underset{at \triangleleft}{\nearrow} & \\ & & Q_2 & & \end{array}$$

Minimal (Q, T)-biautomaton and the hypothesis automaton

We can compute the minimal (Q, T)-biautomaton in an arbitrary category* using off-the-shelf results from (Colcombet,P., 2017).

$$1 \xrightarrow{\triangleright q_{min}} Q/{\sim_{T \cup AT}} \underset{\varepsilon_{min}}{\overset{a_{min}}{\rightrightarrows}} (Q \cup QA)/{\sim_T} \xrightarrow{t\triangleleft_{min}} 2$$

Recall $w \sim_T v$ iff $\forall u \in T. \quad wu \in L \Leftrightarrow vu \in L$

* under mild assumptions

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- ε_{min} is surjective iff (Q, T) is closed
- ε_{min} is injective iff (Q, T) is consistent

$Q = T := \{\varepsilon\}$
repeat
  while (Q, T) not closed and consistent
    if (Q, T) is not closed enlarge Q
      $(\forall q \in Q . \forall a \in A . \exists p \in Q . \ p \sim_T qa)$
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      $(\forall q, q' \in Q . \forall a \in A . \ q \sim_T q' \Rightarrow qa \sim_T q'a)$
  ask an equivalence query for $\mathcal{H}(Q, T)$
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- ε_{min} is surjective iff (Q, T) is closed
- ε_{min} is injective iff (Q, T) is consistent
- If ε_{min} is an isomorphism we merge the two states of the (Q, T)-biautomaton and obtain $\mathcal{H}(Q, T)$.

```
Q = T := {ε}
repeat
  while (Q,T) not closed and consistent
    if (Q,T) is not closed enlarge Q
       (∀q ∈ Q.∀a ∈ A.∃p ∈ Q. p ~T qa)
    if (Q,T) is not consistent enlarge T
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  ask an equivalence query for H(Q,T)
  if the answer is no then
    add the counterexample and its
    prefixes to Q
until the answer is yes
return H(Q,T)
```

The $FunL^*$-algorithm

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input: teacher of the target language L — in a catgeory $\mathcal{C}$
output: $Min(L)$
$Q := T := \{\varepsilon\}$
repeat
  while ε_{min} is not an isomorphism **do** — $\mathrm{Iso} = E \cap M$
    if $\varepsilon_{min} \notin E$ **then** — (E, M) fact. system
      add QA to Q
    if $\varepsilon_{min} \notin M$ **then**
      add AT to T
  ask an equivalence query for the hypothesis automaton $\mathcal{H}(Q, T)$
  if the answer is no **then**
    add the counterexample and all its prefixes to Q
until the answer is yes
return $\mathcal{H}(Q, T)$

Correction and termination of the algorithm

Theorem. Assume $\mathcal{C}$ is a category with a factorization system (E, M), having countable copowers and countable powers.

We consider a target language L in the catgeory $\mathcal{C}$ such that the state space of the minimal automaton for L is (E, M)-noetherian* (generalization of finite).

Then the $FunL^*$-algorithm terminates, eventually producing the minimal automaton $Min(L)$ accepting L.

*(E, M)-noetherianity means no infinite chains of E-quotients or of M-subobjects.

Perspectives

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Some advancement in this direction
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Active Learning of Deterministic Transducers with Outputs in Arbitrary Monoids. CSL 2024.

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And what can be done when we know that $\mathrm{Kl}(T)$ is not good enough?

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Understand this through a lax functor from JSL to Rel... Ongoing work with Quentin Schroeder and Quentin Aristote.

Further extensions

- Extension to tree automata
- Weighted automata over semirings …
- What about other forms of learning, e.g., nominal automata? We can build on Victor Iwaniack's work on automata in toposes.

And even more importantly ...

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SCIENTISTS
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https://tcs4f.org/

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An estimation of the emissions per person for a return trip Paris - Barcelone

- by train : approx. 6 kg CO_2
- by plane : approx. 680 kg CO_2